The Optimal Structure of Securities under Coordination Frictions*

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Abstract

We study multi-agent security design in the presence of coordination frictions. A principal implements a project whose value increases with an unknown state and the level of agents' participation. To encourage participation, the principal offers a bundle of monotone securities backed by the project value. More participation results in a higher project value, making participation decisions strategic complements. Miscoordination arises because agents cannot precisely infer others' decisions from noisy observations about the state. We identify two objects in security design—"payoff sensitivity" and "perception of participation"—that determine the impact of miscoordination. To mitigate the adverse impact of miscoordination, the two objects should be matched assortatively over agents. This mechanism implies multi-tranche structures where senior tranche–holders, who are more robust to miscoordination, participate more aggressively and help alleviate junior tranche–holders' fear of miscoordination. The principal's ability to differentiate agents in security format is crucial to whether differentiation is desirable.

Keywords: multi-agent contracting, coordination frictions, security design, global games, multi-tranche

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1 Introduction

Project implementation often requires multiple agents' participation. For example, an entrepreneur needs to raise capital from multiple investors, a financial system relies on multiple interconnected financial institutions and investors to operate, and a firm organizes multiple divisions to work on joint tasks. In such settings, each agent's payoff is typically contingent on the project outcome, which depends on all agents' participation decisions.[1] As a result, agents care about others' participation decisions, and coordination is crucial. However, coordination can be impeded for various reasons. For instance, coordination through communication may be too costly, especially between a large number of agents; it may take a long time to coordinate while participation decisions are urgent; communication itself may be vague and ineffective due to different interpretations. As a result, the agents face strategic uncertainty among themselves and may miscoordinate in their decisions, making them reluctant to participate. Such reluctance is self-reinforcing, making it even harder for the principal to motivate the agents to participate and achieve efficient outcomes.

This paper studies the principal's optimal security design in the presence of coordination frictions. The principal has the flexibility to design and offer different securities to various agents, allowing for the study of how payoffs are allocated across states and agents, as well as their interactions. The analysis centers on three key questions: What security format is the most effective in addressing miscoordination? Should agents be differentiated in security format or pricing? And what mechanism determines the desirability of a particular way of differentiating the agents?

To address these questions in a unified framework, we formulate the problem in the context of a joint task where a risk-neutral principal (she) aims to develop a project whose value increases in an unknown state and the participation of multiple risk-neutral agents (he).[2] The principal can design a bundle of monotone securities, backed by the project's value, to encourage agents' participation. Increased participation enhances the project value, leading to higher security payments for all agents and making participation decisions strategic complements. Since agents do not observe the state but receive private noisy signals about it, miscoordination may arise due to imperfect inference of others' decisions. As such, we adopt the global games approach and study the principal's multi-agent security design problem on top of it.

In global games equilibria, agents play cutoff strategies: they participate if and only if their signals exceed certain cutoffs. Therefore, understanding the impact of security design on participation requires focusing on marginal agents—those who observe their cutoff signals.

[1] Here, we use the term "participation" in a general sense to refer to an agent's action that contributes to the overall outcome of the project. Depending on the context, this could refer to investing capital, exerting effort, contributing ideas or resources, etc.

[2] We will use she/her to refer to the principal and he/his to (one of) the agents throughout the paper. We don't attribute any significance to these particular gender assignments.

We identify two crucial objects in security design. The first is a marginal agent's *perception of participation*, which represents the probability distribution of the number of participating agents from his perspective. In line with the belief constraint in Sákovics and Steiner (2012), the total perception of participation of all agents is fixed, implying a natural constraint on security design.[3] A marginal agent with a lower (higher) cutoff perceives a lower (higher) likelihood of others' participation, as he expects other agents to receive lower (higher) signals and be less (more) likely to participate. Thus, security design essentially redistributes perception of participation among agents. The second crucial object is an agent's payoff sensitivity, which measures how his expected payoff increases with the participation of others. Since the total payoff offered to agents cannot exceed the project value, this imposes a constraint on the aggregate payoff sensitivity of all agents.

Due to the complementarity in production technology, agents' expected payoffs increase with the expected participation of other agents, allowing the principal to economize on security offerings. We find that the benefits from this complementarity are equal to the product of perception of participation and the payoff sensitivity, summed across all agents. To maximize total benefits while adhering to the aforementioned constraints on total perception of participation and total payoff sensitivity, it is optimal to differentiate agents in both objects and achieve assortative matching between them. This novel finding of an assortative matching mechanism helps us derive the qualitative properties of the optimal security design.

To understand the mechanism, we begin by fixing agents' perception of participation and determining the optimal security formats. We find that the optimal security bundle should be structured in tranches: agents with lower perception of participation receive more senior tranches, while agents with the same perception of participation receive identical tranches. This structure arises because the principal and marginal agents place different importance on cash flows. Firstly, compared with marginal agents, the principal places greater (smaller) importance on security payments when the project value is high (low). Since participating agents are those with signals equal to or higher than their cutoffs, the project value when security payments are made is generally higher than what a marginal agent expects. Consequently, the principal prefers to offer a senior tranche to the agents as a whole and retain the junior tranche for herself. Secondly, marginal agents with lower perception of participation place greater importance on security payments at low project values, as they consider these values more likely. To economize on security offerings, the principal should allocate cash flows at specific project values to agents who value them most, resulting in a multi-tranche structure based on their perception of participation. Since agents with the same per-

[3] When the agents have the same security payment, the subgame of participation becomes a standard global game where the players have symmetric payoffs and play the same cutoff strategy in equilibrium, so that the above constraint reduces to the usual Laplacian belief (i.e., all marginal agents i perceive $f^i(M) = 1$, the uniform distribution over $[0,1]$).

ception of participation value cash flows identically, the principal can offer them identical tranches without loss of efficiency.

Given that the multi-tranche structure is optimal when agents have different perception of participation, we next explore how finely agents should be differentiated in this dimension. We find that finer differentiation is always preferred, and the optimal security bundle should include the maximum number of tranches. Firstly, senior-tranche holders' benefit less from others' participation due to their lower overall payoff sensitivity relative to holders of junior-tranche holders'. Secondly, because the total perception of participation is fixed, a decrease in senior-tranche holders' must be accompanied by an increase in junior-tranche holders'. Thus, reallocating perception of participation from senior-tranche holders to junior-tranche holders benefits the latter more than it harms the former, allowing the principal to reduce the aggregate security offering. In other words, this differentiation creates a beneficial *assortative matching* within the tranching structure. Consequently, the principal should use a multi-tranche structure to differentiate agents such that senior-tranche holders—who are less impacted by miscoordination—participate more aggressively and thereby alleviate the junior-tranche holders' concerns for miscoordination.

Our characterization of the optimal security design provides several new insights into contract design in multi-agent settings. First, the use of multiple tranches, as opposed to a single tranche, can be justified purely by coordination frictions. For comparison, if agents could directly observe the state and coordinate perfectly under the principal's recommendation, offering all agents identical securities would be optimal. However, in the presence of coordination frictions, the optimal security design involves differentiating homogeneous agents to achieve assortative matching, rather than accommodating heterogeneous agents. Thus, the emergence of a multi-tranche structure does not necessarily require heterogeneous agents or a clientele effect.

Second, the desirability of differentiating agents depends on whether it enables an assortative matching between their perception of participation and payoff sensitivity. Differentiation is beneficial because it allows the principal to design security formats flexibly, redistributing the two objects among agents. However, if security formats are restricted, such as when the principal must offer identical formats but can only differentiate agents in pricing, differentiation is not desirable because the restriction prevents the beneficial assortative matching. This point distinguishes our paper from the existing literature on unique implementation (Segal, 2003; Winter, 2004; Halac et al., 2020), where the principal typically prefers to differentiate agents regardless of their payoff structures. To our knowledge, our paper is the first to highlight the significance of differentiating agents' payoff structures in contracting with coordination frictions.

Third, the optimal security design induces differentiation in agents' perception of participation but does not necessarily take it to extremes. Under the optimal design, marginal agents may still face considerable uncertainty regarding the decisions of agents holding different tranches. This indicates that, unlike the literature on unique implementation, the point of differentiation in our model is not to eliminate strategic uncertainty.

The rest of this paper is organized as follows. In the rest of the introduction, we review the related literature. Section 2 illustrates the main insight of our results with a simplified example. Section 3 sets up the formal model. Section 4 conducts the equilibrium analysis for the sub-game of participation for any given security design. Section 5 studies the optimal security design based on the equilibrium analysis in Section 4. Section 6 is devoted to applications and further discussion. All proofs are relegated to the Appendix and Online Appendix.

Related literature

Our paper is related to the literature on global games (Carlsson and van Damme, 1993). Global games have been widely used to model coordination frictions in various settings such as currency attack (Morris and Shin, 1998), debt rollover (Morris and Shin, 2004), and bank run (Goldstein and Pauzner, 2005). Our paper involves differentiation of agents and is particularly related to papers on global games with heterogeneous agents. Frankel et al. (2003) prove limit uniqueness of equilibria for general global games with arbitrary numbers of agents and heterogeneous payoffs. Corsetti et al. (2004) study a game with a single large agent and a continuum of small agents and find that the presence of the large agent makes all other agents participate more aggressively. Unlike these papers, our paper addresses a design problem on top of global games, which demands more detailed characterization of the equilibria. We explicitly formulate agents' perception of participation as functions of the relative distances between agents' equilibrium cutoffs and the distribution of signal noise (See Definition 1). This explicit formulation enables us to examine global games with general payoff structures.[4]

Our paper belongs to the literature on contracting with coordination frictions. The work that is closest to ours is Sákovics and Steiner (2012), who study a principal's optimal subsidies that attain a given likelihood of successful coordination at minimal cost in a coordination game that features ex ante heterogeneous agents and regime-change production technology. Notably, they find that although each agent's perception of participation is not uniform in general due to heterogeneity, the weighted average of theirs is still uniform. This property serves as an important constraint for all kinds of design on top of global games. Our paper differs from theirs in two important ways. First, we do not assume heterogeneous agents but allow differentiation of agents to emerge endogenously in equilibrium. Our discussion centers around in what situations agents should be differentiated and how. In our setup, the principal's optimal design intentionally differentiates homogeneous agents to mitigate the adverse impact of miscoordination, instead of merely responding

[4] Sákovics and Steiner (2012) can explicitly solve for the equilibrium cutoffs without formulating perception of participation because they focus on regime-change payoff structures.

to their exogenous heterogeneity. Second, we address optimal security design in a general framework where the principal can flexibly design different security formats for different agents under a general production technology. This flexibility enables us to identify the key objects that contract design works on and uncover the mechanism of assortative matching between perception of participation and payoff sensitivity, leading to sharp predictions regarding the optimal security formats. In particular, if the agents' payoff sensitivity is fixed and the principal can only shape their perception of participation through subsidies, our model reduces to the setting of Sákovics and Steiner (2012).

Also considering heterogeneous payoffs in global games, Choi (2014), Goldstein et al. (2023), and Dai et al. (2024) find that differentiating agents' payoff structures can help mitigate miscoordination in finance settings. Relatedly, Shen and Zou (2023) examine an intervention program that screens investors based on their interim beliefs in global games. This program enhances investor welfare by compensating those with moderate interim beliefs, thereby increasing their willingness to remain invested and bolstering other investors' confidence in banks. Instead, our model focuses on security design before agents receive private information. Thus, our analysis illuminates enduring contract structures that are independent of interim information, whereas their analysis is more applicable to interim arrangements. Unlike these papers and ours, which consider the unique equilibrium of global games, another strand of literature studies unique implementation in complete-information coordination games, which typically feature multiple equilibria (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Halac et al., 2020, 2021; Hoffmann and Vladimirov, 2023). The requirement for unique implementation steers the attention of contracting to the worst Nash equilibrium, and optimal contracts often employ divide-and-conquer tactics to differentiate agents.

Demonstrating the optimality of multi-tranche structures, our paper is closely related to the literature on security design. Tranching is an prevalent phenomenon in finance and has been rationalized from various perspectives (Myers and Majluf, 1984; Boot and Thakor, 1993; Nachman and Noe, 1994; DeMarzo and Duffie, 1999; Biais and Mariotti, 2005; Demarzo et al., 2005; Axelson, 2007; Yang, 2020). Recently, Frankel (2023) shows that debt is optimal because it minimizes underpricing due to miscoordination in security offerings. The model incorporates a broad formulation of miscoordination that includes several theories from the literature, but multi-tranche structures are not its focus since it does not allow the issuer to offer different securities to investors. Winton (1995) explores multi-investor security design within the classical costly state verification framework, showing that multiple tranches can reduce verification costs by enabling different investors to focus on verifying different subsets of states. Other papers justify multi-tranche structures based on clientele effects, where different tranches cater to heterogeneous agents. [5] In con-

[5] Friewald et al. (2016) indicate that multiple tranches could be favorable due to a postsale clientele effect, where buyers may have heterogeneous holding costs, and multiple tranches can cater to their varying needs for liquidity. Ger-

trast, our paper is the first, to our knowledge, to demonstrate that even absent ex ante heterogeneity, coordination frictions naturally motivate the creation of multi-tranche structures to differentiate ex ante homogeneous agents.

2 An Illustrative Example

This section presents an example to illustrate the paper's main insights. Specifically, we adapt the debt rollover model of Morris and Shin (2004) to our study of security design.

2.1 Setup of the Example

Consider that a firm (i.e., the principal) raises \$1 from each of two banks (i.e., the agents) to develop a project. There are three dates, $t = 0, 1, 2$. All players are risk neutral and do not discount future cash flows. At $t = 0$, the firm designs debt securities and enters into a lending relationship with each bank. Each bank lends \$1 to the firm at $t = 0$ in exchange for a debt payment at $t = 2$. At $t = 1$, each bank receives private information and decides whether to terminate the relationship or not. Termination means forgoing the debt payment and getting the \$1 investment back from the firm immediately. We say a bank participates if he decides not to terminate.

At $t = 2$, the project either succeeds or fails. The firm value will be $\overline{C} > 0$ if the project fails and V if it succeeds. The debt payment to bank i is state-contingent and specified by the debt security (c_i, d_i), where c_i is the payment upon failure and d_i the payment upon success, respectively. We refer to $d_i - c_i$ as bank i's *payoff sensitivity*. The debt security (c_i, d_i) can be interpreted as follows. If the project fails, the firm is worth only the liquidation value of its tangible assets, which is $\overline{C}$, and bank i only receives the tangible assets assigned to him as collateral, which is c_i. When the project succeeds, the firm value is V and the firm is willing to repay the debt face value d_i to avoid forced liquidation. Due to the resource constraint, the total amount of collateral cannot exceed the firm's liquidation value, i.e., $c_1 + c_2 \leq \overline{C}$. We assume $\overline{C} < 1$, i.e., the firm does not have sufficient tangible assets to fully collateralize any loan.

The production technology follows a simple regime change structure. The project's success probability depends on an exogenous state $\theta \in \mathbb{R}$ and the number of participating banks. In state θ, the success probability equals $P_1(\theta)$ if both banks participate and $P_0(\theta)$ otherwise. Both $P_0(\theta)$ and $P_1(\theta)$ are continuous and increasing in θ. We assume $P_1(\theta) > P_0(\theta)$. That is, the project is more likely to succeed when more banks participate. We also assume the existence of dominance regions so that we can apply the global game approach: $P_0(\theta)$ and $P_1(\theta)$ go to 1 as $\theta \to +\infty$ and

shkov et al. (2023) shows that tranching can endogenously arise in an optimal mechanism because of the differences in risk appetites and budget constraints among agents.

to 0 as $\theta \to -\infty$.

The firm and the banks share a common prior of θ at $t=0$. At $t=1$, before deciding whether to participate, each bank i observes a private signal $x_i = \theta + \sigma\varepsilon_i$. All ε_i follow the cumulative distribution function $\Phi(\cdot)$ (with probability density function $\phi(\cdot)$) and is independent conditional on θ. σ represents the magnitude of the noise. To focus on the implications of coordination frictions for security design, we aim to derive optimal securities in the limit case of $\sigma \to 0$.

2.2 Banks' Participation Decisions

Since the banks' initial investment is refundable at $t=1$ and they do not discount the cash flows, they always enter into the lending relationship at $t=0$. We thus only need to consider the banks' participation decisions at $t=1$. Given the securities, banks play a standard global game with heterogeneous payoffs. We follow Sákovics and Steiner (2012) and Dai et al. (2024) to characterize the equilibrium of the game. As is well understood in the literature, for sufficiently small σ, the game has a unique equilibrium in which each bank i participates if and only if his signal x_i is greater than a cutoff $\hat{x}_i$.[6] We define bank i's absolute distance as the other one's cutoff minus his, $\hat{x}_j - \hat{x}_i$, and define his relative distance as his absolute distance scaled by the magnitude of the noise, $\Delta_i \triangleq \left(\hat{x}_j - \hat{x}_i\right)/\sigma$. By construction, banks' relative distance must be opposite: $\Delta_1 = -\Delta_2$.

We regard a bank as marginal if he observes his cutoff signal. A marginal bank i is indifferent to participating or not, so the following indifference condition holds:

$$c_i + (d_i - c_i)\cdot p_i = 1,$$

where

$$p_i = \int_{-\infty}^{\infty} \left\{ P_0(\theta) + [P_1(\theta) - P_0(\theta)] \left[1 - \Phi\left(\frac{\hat{x}_j - \theta}{\sigma}\right)\right] \right\} \frac{1}{\sigma} \phi\left(\frac{\hat{x}_i - \theta}{\sigma}\right) d\theta$$

represents bank i's expected success probability. In the limit case of $\sigma \to 0$, p_i has an intuitive expression as follows:

$$p_i = p\left(\hat{x}_i, \Delta_i\right) \triangleq P_0\left(\hat{x}_i\right) + \left[P_1\left(\hat{x}_i\right) - P_0\left(\hat{x}_i\right)\right] \cdot f\left(\Delta_i\right), \tag{1}$$

where

$$f\left(\Delta_i\right) = 1 - \int_0^1 \Phi\left(\Phi^{-1}(1-y) + \Delta_i\right) dy. \tag{2}$$

Since the signal is very precise about the state, fundamental uncertainty vanishes. Marginal bank i thinks the state θ is equal to $\hat{x}_i$ almost surely. The project's success probability is $P_0(\hat{x}_i)$ if

[6] Here we assume that bank i does not participate if he observes $x_i = \hat{x}_i$. This tie-breaking rule is without loss of generality, as observing $x_i = \hat{x}_i$ is a zero-probability event.

the other bank does not participate, and $P_1(\hat{x}_i)$ if he does. However, strategic uncertainty can remain substantial. Marginal bank i is not always sure about the other's participation decision. The probability he perceives is denoted by $f(\Delta_i)$, which is a function of his relative distance.

We refer to $f(\Delta_i)$ as bank i's *perception of participation*. Figure 1 graphically illustrates $f(\Delta_i)$. The top panel plots the posterior distribution of x_2 perceived by bank 1 upon observing $\hat{x}_1$, $g(x_2 \mid \hat{x}_1)$. $f(\Delta_1)$ is the probability that x_2 exceeds $\hat{x}_2$, and thus represented by Area 1. It is easy to see that Area 1 depends on the relative distance between $\hat{x}_2$ and $\hat{x}_1$ relative to the dispersion of the distribution. Hence, it is a decreasing function of bank 1's relative distance. The bottom panel of Figure 1 plots both banks' posterior distributions upon observing their respective cutoffs. $f(\Delta_2)$ is the probability that x_1 exceeds $\hat{x}_1$, and thus represented by $1-$ Area 2. Since the two posterior distributions exhibit mirror symmetry,[7] Area 1 and Area 2 in the graph are equal. Hence, the two banks' total perception of participation is always 1:

$$f(\Delta_1) + f(\Delta_2) = \text{Area } 1 + (1 - \text{Area } 2) = 1. \tag{3}$$

First illustrated by Sákovics and Steiner (2012), this identity stems from Bayes' rule when agents' signals are very informative and generally holds for multiple agents. It follows the same logic as the usual Laplacian belief in the global game models and nests it as a special case with homogeneous players. In that case, the two banks' cutoffs coincide, i.e., $\Delta_1 = 0$, so that upon observing the common cutoff signal, each bank believes that the other bank participates with probability 0.5. Since each bank's expected success probability depends on his perception of participation (equation (1)), this identity implies a budget constraint on banks' expected success probabilities. The following lemma summarizes important properties of perception of participation.

Lemma 1. *$f(\Delta_i)$ is nonnegative, strictly decreasing in Δ_i, and $f(\Delta_1) + f(\Delta_2) = 1$.*

In the limit case of $\sigma \to 0$, banks' indifference conditions are written as follows:

$$c_1 + (d_1 - c_1) p(\hat{x}_1, \Delta_1) = 1, \tag{4}$$

$$c_2 + (d_2 - c_2) p(\hat{x}_2, \Delta_2) = 1. \tag{5}$$

Suppose that bank 1's cutoff is weakly lower, so $\hat{x}_1 \leq \hat{x}_2$ and $\Delta_1 \geq 0$. The equilibrium takes two possible forms. When the two securities are close in attractiveness, their cutoffs are very close. As $\sigma \to 0$, banks' absolute distances converge to 0, but their relative distances remain substantial. That means, $\hat{x}_1 = \hat{x}_2$ and $\Delta_1 = -\Delta_2 \in [0, +\infty]$. In this case, marginal banks have almost the same view on the state θ, but different views on the other's participation. When the two securities are quite different in attractiveness, as $\sigma \to 0$, banks' absolute distances remain substantial, and their

[7] This feature holds even when banks' signal noises follow different distributions.

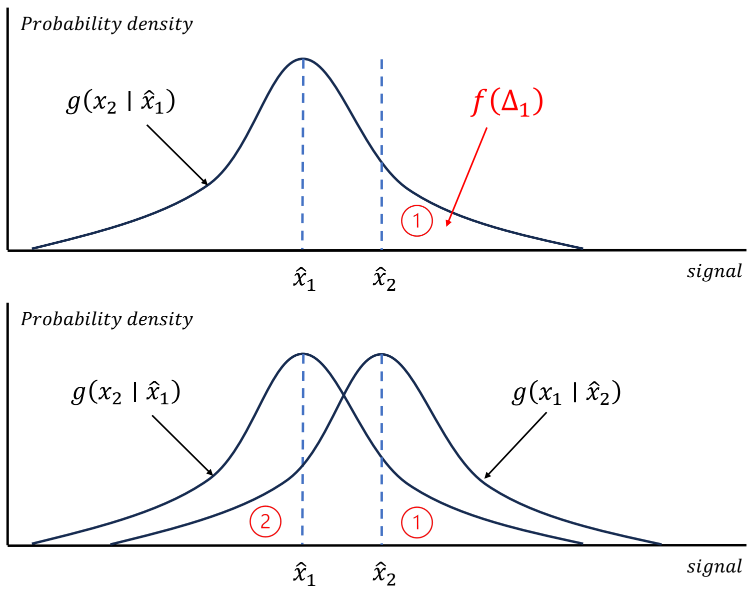

Figure 1: The probability distribution of the other bank's signal from a marginal bank's perspective

relative distances go to infinity. That means, $\hat{x}_1 < \hat{x}_2$ and $\Delta_1 = -\Delta_2 = +\infty$. In this case, marginal banks have different views on both the state θ and the other's participation.

2.3 Assortative Matching

In this multi-agent security design problem, we are particularly interested in whether and how agents should be differentiated. We find that the core mechanism underlying this design question can be summarized as assortative matching between payoff sensitivity and perception of participation. This subsection illustrates this mechanism by comparing symmetric and asymmetric securities.

As the benchmark, symmetric securities (c,d) are offered, and the two banks have the same cutoff, say $\hat{x}$. As the alternative, we consider a class of securities $\{(c_i,d_i)\}_{i=1,2}$ such that the following equation system holds for some nonnegative Δ:

$$c_1 + (d_1 - c_1)p(\hat{x},\Delta) + c_2 + (d_2 - c_2)p(\hat{x},-\Delta) = 2,$$
$$c_1 + (d_1 - c_1)p(\hat{x},\Delta) = c_2 + (d_2 - c_2)p(\hat{x},-\Delta),$$
$$c_1 + c_2 = 2c.$$

The first two equations say that banks' expected payoffs at $\hat{x}$ are equal and their sum is 2. They are equivalent to banks' indifference conditions, equations (4) and (5), at $\hat{x}$. They imply that under the alternative, the two banks' cutoffs are $\hat{x}$, bank 1's relative distance is Δ, and bank 2's is $-\Delta$. The third equation says that the alternative always takes up the same total amount of collateral as the benchmark.

It is not hard to see that under both the benchmark and the alternative, each bank participates when $\theta > \hat{x}$ and otherwise does not participate. That means, they bring the same amount of capital to the firm in almost all states, and the total payment to banks is the aggregate security when $\theta > \hat{x}$ and 0 otherwise. Since they take up the same total amount of collateral, the firm prefers the one with the lower total face value. Hence, we want to find an alternative to minimize the total face value subject to the three equations. The problem is formulated as follows:

$$\begin{aligned}\min_{(c_1,d_1),(c_2,d_2),\Delta} \quad & d_1 + d_2 \\ \text{s.t.} \quad & c_1 + (d_1 - c_1) p(\hat{x},\Delta) + c_2 + (d_2 - c_2) p(\hat{x},-\Delta) = 2, \\ & c_1 + (d_1 - c_1) p(\hat{x},\Delta) = c_2 + (d_2 - c_2) p(\hat{x},-\Delta), \\ & c_1 + c_2 = 2c.\end{aligned}$$

To see what securities in this class can have $d_1 + d_2 < 2d$, we can instead consider the dual problem. Instead of minimizing the total face value, we fix it to $2d$ and maximize the sum of banks' expected payoffs at $\hat{x}$. If there exist securities such that the sum of banks' expected payoffs at $\hat{x}$ is greater than 2, then the firm can reduce d_1 and d_2 and still make two banks indifferent at $\hat{x}$ in equilibrium. After leaving out constant terms, the objective becomes the product of payoff sensitivity and expected success probability summed over banks.

$$\begin{aligned}\max_{(c_1,d_1),(c_2,d_2),\Delta} \quad & (d_1 - c_1) p(\hat{x},\Delta) + (d_2 - c_2) p(\hat{x},-\Delta) \\ & c_1 + c_2 = 2c, \\ & d_1 + d_2 = 2d, \\ & c_1 + (d_1 - c_1) p(\hat{x},\Delta) = c_2 + (d_2 - c_2) p(\hat{x},-\Delta),\end{aligned} \tag{6}$$

What can we do to increase this objective? We cannot change the total payoff sensitivity, since it equals the payoff sensitivity of the aggregate security:

$$(d_1 - c_1) + (d_2 - c_2) = 2d - 2c.$$

We cannot change the total expected success probability. It is constant since the total perception of participation equals 1 according to Lemma (1):

$$\begin{aligned}p(\hat{x},\Delta)+p(\hat{x},-\Delta) &= 2P_0(\hat{x})+[P_1(\hat{x})-P_0(\hat{x})]\cdot[f(\Delta)+f(-\Delta)] \\ &= P_0(\hat{x})+P_1(\hat{x}).\end{aligned}$$

In maths, we can only increase the objective by strengthening the *assortative matching* between *payoff sensitivity* and *perception of participation*. That is, we want bank 1 to have both the lower payoff sensitivity and the lower perception of participation, and bank 2 to have both the higher payoff sensitivity and the higher perception of participation. Security design can directly determine payoff sensitivity and can affect perception of participation through the condition that banks' expected payoffs at $\hat{x}$ are equal, which is equation (6). The following two examples illustrate how security design potentially affects the assortative matching.

Example 1. Under the benchmark contract, both banks are offered (c,d). They have the same payoff sensitivity $d-c$ and the same perception of participation $f(0)$. Hence, the objective equals

$$(d-c)\,p(\hat{x},0)+(d-c)\,p(\hat{x},0)=(d-c)\,(P_1(\hat{x})+P_0(\hat{x}))\,.$$

Now we slightly increase bank 1's collateral and decrease bank 2's by α, while keeping their debt face value unchanged. This change has two effects. First, bank 1's payoff sensitivity is lower than bank 2's: $d-(c+\alpha)<d-(c-\alpha)$. Second, bank 1's security is more attractive than bank 2's, so bank 1's perception of participation is lower than bank 2's, according to equation (6). Taken together, this change strengthens the assortative matching and increases the objective

$$\begin{aligned}[d-(c+\alpha)]\,p(\hat{x},\Delta)+[d-(c-\alpha)]\,p(\hat{x},-\Delta) &= (d-c)\,(P_1(\hat{x})+P_0(\hat{x}))+\frac{\alpha^2}{d-c}\,(2-P_1(\hat{x})-P_0(\hat{x})) \\ &> (d-c)\,(P_1(\hat{x})+P_0(\hat{x}))\,.\end{aligned}$$

Example 2. We can also slightly increase bank 1's debt face value and decrease bank 2's by α, while keeping their collateral unchanged. Bank 1 still has the lower perception of participation due to a more attractive security, but it also has the higher payoff sensitivity. This change weakens the assortative matching and decreases the objective

$$\begin{aligned}[(d+\alpha)-c]\,p(\hat{x},\Delta)+[(d-\alpha)-c]\,p(\hat{x},-\Delta) &= (d-c)\,(P_1(\hat{x})+P_0(\hat{x}))-\frac{\alpha^2}{d-c}\,(P_1(\hat{x})+P_0(\hat{x})) \\ &< (d-c)\,(P_1(\hat{x})+P_0(\hat{x}))\,.\end{aligned}$$

The two examples suggest that concentrating collateral on one bank can be desirable because it causes a bank's payoff sensitivity and perception of participation to change in the same direction, but concentrating face value on one bank may not because it causes changes in opposite directions. These forces will determine whether and how optimal securities differentiate the two banks.

2.4 Optimal securities

Next, we derive optimal securities in different situations and discusses them from the perspective of assortative matching.

Proposition 1. *Under the optimal securities, the two banks have a common cutoff in the limit case. Denote it by* $\hat{x}$. *If Firm can offer any combination of securities, the optimal securities constitute a multi-tranche structure in which 1)* $c_1 = \overline{C}$, $c_2 = 0$, *2)* $d_1 - c_1 < d_2 - c_2$, *and 3)* $f(\Delta_1) < f(\Delta_2)$. *Specifically,*

- *If* $\overline{C} < 1 - \left(\frac{P_0(\hat{x})}{P_1(\hat{x})}\right)^2$, $d_1 - c_1 = \frac{1-\overline{C}+\sqrt{1-\overline{C}}}{P_0(\hat{x})+P_1(\hat{x})}$, $d_2 - c_2 = \frac{1+\sqrt{1-\overline{C}}}{P_0(\hat{x})+P_1(\hat{x})}$, *and* $0 < f(\Delta_1) < f(\Delta_2) < 1$.
- *If* $\overline{C} \geq 1 - \left(\frac{P_0(\hat{x})}{P_1(\hat{x})}\right)^2$, $d_1 - c_1 = \frac{1-\overline{C}}{P_0(\hat{x})}$, $d_2 - c_2 = \frac{1}{P_1(\hat{x})}$, $f(\Delta_1) = 0$, *and* $f(\Delta_2) = 1$.

Due to the production technology we assume here, optimal securities must result in the two bank having a common cutoff in the limit case.[8] Proposition 1 states that if the firm can offer any combination of securities, the optimal securities constitute a multi-tranche structure in which bank 1 is the senior-tranche holder, who is assigned all the collateral and has both the lower *payoff sensitivity* and the lower *perception of participation*; bank 2 is the junior-tranche holder, who is assigned no collateral and has both the higher *payoff sensitivity* and the higher *perception of participation*. The optimality of the multi-tranche structure comes from two observations. The first concerns how to allocate cash flows between banks. Assortative matching implies that we should concentrate collateral on one bank. This observation is driven by the coordination friction induced by incomplete information. The second concerns how to cash flows between the firm and banks. The firm should give banks the senior part. Section 5.2 discusses this point in detail.[9] This observation is driven by classic adverse selection. It appears in many settings and does not rely on coordination frictions particularly. Therefore, we regard the result on how to allocate cash flows between banks as the most important, as it delivers the unique economics about coordination frictions.

[8] The reason is that the success probability increases from $P_0(\theta)$ to $P_1(\theta)$ only when both banks participate. In any state θ, the firm would prefer no bank participating to only one bank participating, because it pays less to banks in the former case. Therefore, securities that result in different cutoffs in the limit case cannot be optimal.

[9] The short intuition is as follows. Marginal banks care about the value of a security when signals equal cutoffs. The firm needs to pay the value of a security to banks that participates or in other words, whose signals are equal to or greater than their cutoffs. Hence, marginal banks care about low states more than the firm. Therefore, instead of cash flows in high states, the firm is more willing to give banks the cash flows in low states, which constitute the senior part.

Besides the multi-tranching structure, some other features of the optimal securities are worth noting. First, the benefit of differentiation is substantial and does not vanish as $\sigma \to 0$. Actually, if the firm wants to use identical securities to achieve the common cutoff $\hat{x}$, it needs to offer each bank

$$(c,d) = \left(\frac{\overline{C}}{2}, \frac{2-\overline{C}}{P_0(\hat{x}) + P_1(\hat{x})} + \frac{\overline{C}}{2}\right)$$

and gets substantially worse off than offering the optimal securities. Second, strategic uncertainty may remain substantial under the optimal securities. When $\overline{C} < 1 - \left(\frac{P_0(\hat{x})}{P_1(\hat{x})}\right)^2$, $f(\Delta_1)$ is strictly between 0 and 1. That means, the two banks are not clear about each other's decision upon observing their respective cutoffs. The intuition is that to eliminate strategic uncertainty and push $f(\Delta_1)$ to 0, the firm needs to make bank 1's security sufficiently more attractive than bank 2's. Given that the total collateral is not large, the firm needs to make bank 1's face value large. However, the second example suggests that concentrating face value on one bank can weaken the assortative matching, which is not desirable. Hence, the firm prefers not to push $f(\Delta_1)$ to 0. This feature of the optimal securities manifests that the security design here is not intended to eliminate strategic uncertainty as in the literature on unique implementation (Segal, 2003; Winter, 2004; Halac et al., 2020). In that literature, strategic uncertainty needs to be eliminated so that agents perceive the complementarity of others' participation.

In some settings, payoff structures may be restricted in certain ways, making the desirable multi-tranching structure unavailable. Some exercises in the literature can be considered as restricted security design in our framework (Sákovics and Steiner, 2012; Choi, 2014; Dai et al., 2024). A kind of restricted security design that is common in practice is that only securities with the same format are offered. For example, in a syndicated loan, the firm usually offers loans with the same format to banks and can differentiate them through upfront fees. This amounts to imposing a collinearity constraint on banks' payoffs, i.e., $c_2/c_1 = d_2/d_1$. With the assortative matching mechanism, we can easily see that the optimal securities are identical securities.

Proposition 2. *If the firm can offer only collinear securities, the optimal securities are identical securities.*

Due to collinearity, the bank with the lower perception of participation must have the higher payoff sensitivity. If c_1 is greater than c_2, then d_1 is greater than d_2, so bank 1 has the more attractive security and thus the lower perception of participation. However, bank 1 also has the higher payoff sensitivity: $d_1 - c_1 > d_2 - c_2$. Hence, differentiation always weakens the assortative matching between *payoff sensitivity* and *perception of participation*, and is not desirable. This case shows that whether differentiation is preferable depends on whether agents' payoff structures can be differentiated in certain ways. This point helps contrast our paper with the literature on unique implementation (Segal, 2003; Winter, 2004; Halac et al., 2020), where differentiation that eliminates strategic uncertainty is always preferred irrespective of agents' payoff structures. In Section 5.4, we discuss this contrast in more detail.

3 The Formal Model

This section sets up the formal model with multiple agents and general production technologies. We formalize it as a joint task problem with one principal (she) and a continuum of agents (he) of unit mass indexed by $i \in [0,1]$.[10] The principal has a project and wants to motivate agents to participate in the project by offering contracts. All players are risk-neutral and do not discount the future cash flows.

3.1 The Production Technology

Again, let $\theta \in \mathbb{R}$ denote the exogenous fundamental state. The principal and the agents share a common prior of θ, characterized by a probability density function $h(\theta)$ and the corresponding cumulative distribution function $H(\theta)$. We assume that $h(\theta)$ is continuous, bounded, and fully supported on $(-\infty,+\infty)$. Given the fundamental state θ and the mass of participating agents K, the project value z follows a p.d.f. $g(z;\theta,K)$ and a c.d.f. $G(z;\theta,K)$. To ensure that the principal-agent problem is well defined, we assume that the expected project value conditional on θ and K, $E[z|\theta,K]$, and that conditional on K, $E[z|K]$, are always finite. We make the following assumptions regarding the production technology.

Assumption 1.

1. *If* $(\theta_1,K_1) \geq (\theta_2,K_2)$ *and* $(\theta_1,K_1) \neq (\theta_2,K_2)$, *then* $g(z;\theta_1,K_1)/g(z;\theta_2,K_2)$ *is strictly increasing in* z.

2. *For all* $\theta \in \mathbb{R}$ and $K \geq 0$, *the support of* $g(z;\theta,K)$ *is* $[0,+\infty)$, and $G(z;\theta,K)=0$.

The first condition states that $g(z;\theta,K)$ satisfies the strict monotone likelihood ratio property (SMLRP). The higher the fundamental state and the participation of agents, the more likely the project will have a higher value. As an immediate implication of this assumption, the expected project value is strictly increasing in θ and K—i.e., $E[z|\theta_1,K_1] > E[z|\theta_2,K_2]$ if $(\theta_1,K_1) \geq (\theta_2,K_2)$ and $(\theta_1,K_1) \neq (\theta_2,K_2)$. The second condition means that the project generates positive cash flows almost surely but cannot generate any certain positive cash flow.

[10] We will use she/her to refer to the principal and he/his to the agents throughout the article. We don't attribute any significance to these particular gender assignments.

We allow $g(z;\theta,K)$ to have multiple but a finite number of regime changes. Specifically, for $i \in \{0,1,\ldots,J-1,J\}$,

$$g(z;\theta,K) = g_i(z;\theta,K), \text{if } \lambda_i(\theta) \leq K < \lambda_{i+1}(\theta),$$

where $\lambda_0(\theta) = -\infty$, $\lambda_{J+1}(\theta) = +\infty$, and $\lambda_i(\theta)$ is weakly decreasing in θ and strictly increasing in i for $i \in \{1,\ldots,J-1,J\}$. Except for the points where a regime change occurs, the expectation of z is Lipschitz continuous in θ and K. Specifically, there exists $R > 0$ such that for all i,

$$|E_i[z|\theta_1,K_1] - E_i[z|\theta_2,K_2]| \leq R|\theta_1 - \theta_2| + R|K_1 - K_2|,$$

where $E_i[z|\theta,K] \triangleq \int_0^{+\infty} z g_i(z;\theta,K)\,dz$.

3.2 Contracting

The principal can choose a subset of agents and make a take-it-or-leave-it offer to each of them. An offer is a security that specifies the payment to an agent if he participates.[11] As is standard in the literature (Segal, 2003; Winter, 2004; Sákovics and Steiner, 2012; Halac et al., 2020), we assume that the principal can rely on only bilateral contracts. That is, the payment to an agent does not depend on other agents' participation decisions except insofar as those decisions affect the project value. This restriction to bilateral contracts is not trivial and does not fit all cases in practice, but there are considerable cases it fits. In many settings, it is difficult to verify agents' participation, so only bilateral contracts can be well implemented. For example, employee salaries and bonuses typically depend on only the firm or the team's performance. In some other settings, contracts are nominally multilateral but become de facto bilateral due to strategic actions. For example, when only tangible assets are pledgeable, a creditor's value in bankruptcy is basically the value of the tangible assets assigned to him as collateral, which does not depend on other creditor's decision. We further assume that the project value z is verifiable but the fundamental state θ is not. Hence, the security offered to each agent i is a function of the project value z, $s^i[z]$.

Upon receiving an offer, an agent either participates in the project at the opportunity cost of 1 or rejects the offer. Throughout the paper, unless otherwise specified, "the agents" refer to those who receive offers. The principal's payoff equals the project value minus the total payment to the participating agents. We assume that the principal can offer no more than $N \geq 1$ types of securities.[12] Suppose that the kth type security $s_k[\cdot]$ is offered to a measure Q_k of agents. We refer to these agents as type-k agents. Define $S_k[\cdot] \equiv Q_k s_k[\cdot]$ to represent the aggregate security offered

[11] Potentially, the principal can offer a menu of securities to an agent but she does not benefit from it. Please see Section 6 for more details.

[12] Although N is fixed as a parameter, it is allowed to take any positive integer value.

to type-k agents. Then, $\{(S_k, Q_k)\}_{k=1}^n$ represents the security bundle designed by the principal.[13] We impose the following feasibility constraints on the security design.

Assumption 2. *The securities should satisfy the following conditions:*

1. *(Budget Constraint) The aggregate security payment cannot exceed the project value, i.e,* $\sum_{k=1}^n S_k[z] \leq z$.
2. *(Nonnegativity and Monotonicity) For any type k,* $S_k[z]$ *is nonnegative and weakly increasing in* z.
3. *(Nontriviality) For any type k, there exists a* $z_k > 0$*, such that* $s_k[z] > 1$ *for all* $z > z_k$.
4. *(Lipschitz Continuity) For any type k,* $S_k[z]$ *is Lipschitz continuous in* z*. That is, there exists* $R_k > 0$ *such that* $|S_k[z_1] - S_k[z_2]| \leq R_k |z_1 - z_2|$.

The first condition requires that the total payment to agents does not exceed the project value. This budget constraint ensures that the principal can always fulfill her financial commitment to the agents. The second condition requires the securities be nonnegative, since the agents are protected by limited liability. It also imposes the monotonicity constraint on the securities, which is a usual assumption in the literature and well microfounded by the "side loan" argument as in Innes (1990). The monotonicity constraint together with the SMLRP of $g(z;\theta,K)$ implies the strict monotonicity of the expected payment in θ and K, i.e., $E[S_k[z]|\theta_1,K_1] > E[S_k[z]|\theta_2,K_2]$ if $(\theta_1,K_1) \geq (\theta_2,K_2)$ and $(\theta_1,K_1) \neq (\theta_2,K_2)$. The third condition requires that the security payment $s_k[z]$ not be uniformly below 1, the opportunity cost of the agents. Otherwise, the type-k agents will always reject the offer and thus it is equivalent to removing these agents from I. The fourth condition imposes the Lipschitz continuity of the security payment in the project value. The Lipschitz continuity can be implied by the dual monotonicity constraint usually assumed in the security design literature and thus is a weaker condition.[14] With Lipschitz continuity, we obtain the following lemma regarding how the expected value of a security vary with θ and K.

Lemma 2.

$$|E[S_k[z]|\theta_1,K_1] - E[S_k[z]|\theta_2,K_2]| \leq R_k |E[z|\theta_1,K_1] - E[z|\theta_1,K_2]| + R_k |E[z|\theta_1,K_2] - E[z|\theta_2,K_2]|.$$

Finally, we use $T_F[z] \triangleq \min\{z,F\}$ to represent a senior tranche with face value F. For $0 < F_1 < F_2$, we call $T_{F_2} - T_{F_1}$ a junior tranche relative to T_{F_1}.

[13] Since agents are homogeneous when the security bundle is offered, how different securities are allocated among agents does not matter to the principal. She can offer each agent a lottery, the outcome of which specifies the security for that agent.

[14] The dual monotonicity constraint requires that the residual cash flow received by the principal be also weakly increasing, i.e., $z - \sum_{k=1}^n S_k[z]$ is weakly increasing in z.

3.3 The Timeline and Information

The model has three dates. At date 0, the principal offers a security bundle to a subset of agents. At date 1, nature draws a value of θ, and each agent i observes a noisy signal of θ, $x^i = \theta + \sigma\varepsilon^i$. As in Section 2, ε^i follows cumulative distribution function $\Phi(\cdot)$ (with probability density function $\phi(\cdot)$) and is independent of the state θ and noise ε^j for $j \neq i$; the term σ captures the magnitude of the noise. Upon observing his signal x^i, agent i decides whether to participate in the project. At date 2, the project value is realized. Participating agents receive their respective security payments, and the principal receives the residual value of the project. We assume that p.d.f. ϕ is continuous and fully supported on $(-\infty, +\infty)$.[15] We also require ϕ to satisfy the SMLRP—i.e., for any $a > 0$, $\phi(\varepsilon)/\phi(\varepsilon + a)$ is strictly increasing in ε. Then we obtain the following lemma regarding $\phi(\cdot)$.

Lemma 3. *$\phi(\cdot)$ is bounded, and* $\lim_{|\varepsilon|\to+\infty}\phi(\varepsilon) = 0$.

3.4 Security Design under Coordination Frictions

Agents' decisions are strategic complements in this game because each agent's security payment weakly increases in the project value, which in turn strictly increases in other agents' participation. This creates motives for an agent to coordinate his decision with others at date 1. However, since agents only observe noisy signals, they know neither the state nor the mass of participating agents exactly. The subgame at date 1 is thus a coordination game with incomplete information, which is essentially a global game with agents' (potentially heterogeneous) payoffs designed by the principal at date 0. As is standard in the global game literature (Frankel et al., 2003), we introduce the following technical assumption about the production technology to ensure that there exists a unique equilibrium of the subgame.

Assumption 3. *As $\theta \to -\infty$, $E[z|\theta, 1] \to 0$. For any $\hat{z} > 0$, as $\theta \to +\infty$, $G(\hat{z}; \theta, 0) \to 0$.*

This assumption requires that the project's expected value vanishes when the fundamental state is sufficiently weak, even though all agents participate; and that the project's value can be arbitrarily large when the fundamental state is sufficiently strong, even though no agent participates.[16] As an implication, for any given finite-type security bundle, the subgame at date 1 has dominance regions and hence the contagion argument of the global game approach works. Lemma 4 below formalizes this implication.

[15] We make this full support assumption for technical convenience. Under certain conditions, our main results hold for bounded support as well.

[16] Assumption 3 is not stringent, as it allows $E[z|\theta, 1]$ and $G(\hat{z}; \theta, 0)$ to go to 0 arbitrarily slowly. It is satisfied by many popular probability distributions on $\mathbb{R}^+$, such as the lognormal distributions $\ln z \sim N(\theta + K, \sigma^2)$ and the exponential distribution with rate parameter $e^{-(\theta+K)}$. It nicely ensures the uniqueness of the equilibrium but otherwise does not affect our results qualitatively.

Lemma 4. *There exists a* $\underline{\theta}$ *and* $\overline{\theta}$ *such that for all* $K \geq 0$*,* $E[s_k[z]|\theta,K] - 1 < 0$ *for* $\theta \leq \underline{\theta}$ *and* $E[s_k[z]|\theta,K] - 1 > 0$ *for* $\theta \geq \overline{\theta}$*.*

As is standard in the global game literature, we study the limit case of $\sigma \to 0$. This allows us to focus on the strategic uncertainty, which gives rise to the miscoordination risk, rather than the fundamental uncertainty about θ.

3.5 The summary of the main results

Our analysis mainly addresses two questions. First, should the agents be differentiated in the presence of coordination frictions, and if so, how? This question is central to the literature on contracting with coordination frictions. Second, what security format should be used? This question is central to the literature on security design. We briefly summarize the main results here.

We consider symmetric equilibria in which all agents of the same type play the same strategy. In the limit case, for each type k, a type-k agent participates if and only if he observes a signal greater than a cutoff $\hat{x}_k$ (Proposition 3). Potentially, a security bundle may induce agents to have different limiting cutoffs. Suppose that the set of all limiting cutoffs has T distinct elements, say $\theta_1 < \theta_2 < \ldots < \theta_T$. In addition, for notational convenience, we let $-\theta_0 = \theta_{T+1} = +\infty$. Let K_t be the mass of agents whose cutoff is no more than θ_t, i.e., $K_t = \sum_{\{k|\hat{x}_k \leq \theta_t\}} Q_k$. Then the mass of participating agents is essentially a step function of the state θ:

$$K(\theta) \equiv \sum_{t=1}^{T} K_t \cdot 1\{\theta_t < \theta \leq \theta_{t+1}\} \tag{7}$$

(Proposition 4). We refer to $\{K_t, \theta_t\}_{t=1}^{T}$ as a T-cutoff participation scheme.

Since the expected project value is completely determined by the participation scheme, the principal's problem can be divided into two steps. The first step is to pick a participation scheme. The second step is to minimize the total expected payment to agents, given that the participation scheme is implemented. Suppose the optimal participation scheme implemented by no more than N types of securities induces T^* different limiting cutoffs and contains N^* types of securities. Our main results characterize N^* as follows (Propositions 6, 7, and 8).

Theorem 1. *If the principal can offer any security bundle,* $N^* = N$*. The optimal security bundle satisfies*

- *agents of different types have different perception of participation, i.e.,* $\Delta_{k-1,k} > 0$ *for all* $k \in \{2, 3, \ldots, N-1, N\}$*;*
- *agents with lower perception of participation are offered more senior tranches, i.e.,* $S_k = T_{F_k} - T_{F_{k-1}}$*, where* $0 = F_0 < F_1 < \ldots < F_N$*.*

If the principal is restricted to offering a security bundle with collinear securities, $N^* = T^*$. The optimal security bundle must offer the same security to agents with the same limiting cutoff.

In general, T^* depends on the production technology and can be any integer between 1 and N.[17] Implementing a participation scheme with T^* cutoffs entails at least T^* types. If the principal is flexible in differentiating agents in security format, she strictly benefits from differentiating agents in a finer manner through multi-tranching since it achieves stronger assortative matching. She uses all N types by offering different securities to agents with the same limiting cutoff. However, if the principal is not flexible in differentiating agents in security format, offering different securities achieves weaker assortative matching. Hence, she offers the same security to agents with the same limiting cutoff and uses only T^* types.

4 The Equilibrium Following Any Security Offering

We solve the security design problem by backward induction. This section derives the unique equilibrium of the subgame at date 1 following any security offering.

4.1 The equilibrium characterization

Suppose that a bundle of n types of securities is offered at date 0. The subgame at date 1 is a global game in which agents have potentially heterogeneous payoffs. Consider a symmetric equilibrium in which for each type k in $\{1, 2, \ldots, n\}$, each type-k agent participates if and only if he observes $x \geq \hat{x}_k^\sigma$.[18]

Let $m_k^\sigma(\theta)$ be the probability that a type-k agent participates if the state is θ. Then

$$m_k^\sigma(\theta) = Pr[x \geq \hat{x}_k^\sigma | \theta] = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right). \tag{8}$$

As is usual in models with a continuum of players, we adopt the law of large numbers convention[19] so that the mass of the participating agents is

$$M^\sigma(\theta) \equiv \sum_{k=1}^{n} Q_k m_k^\sigma(\theta) = \sum_{k=1}^{n} Q_k - \sum_{k=1}^{n} Q_k \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right), \tag{9}$$

[17] T^* can be as low as 1. When agents' participation does not generate positive surplus unless it triggers a regime change, for instance, as in the simplified example, it is optimal to make agents' limiting cutoffs coincide with the state where the regime change occurs.

[18] As is well known in the global games literature, it is without loss of generality to focus on symmetric equilibria with switching strategies when the noise is small.

[19] The law of large numbers is not well defined for a continuum of random variables (Sun, 2006). Our convention is equivalent to assuming that the agents' play is the limit of play of finite selections from the population.

which is strictly increasing in θ. As before, we refer to an agent as marginal if he observes his cutoff signal. A marginal type-k agent must break even in expectation, so

$$\int_{-\infty}^{\infty}\left[\int_{0}^{\infty}(s_k[z]-1)\,g\left(z;\theta,M^{\sigma}(\theta)\right)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)h(\theta)d\theta=0. \tag{10}$$

It is worth noting that the posterior probability of θ contains two sources of information: one is the prior $h(\theta)$, and the other is the agent's private signal $x^i=\hat{x}_k^{\sigma}$.

For small σ, the private signal is sufficiently accurate relative to the prior information. We can thus simplify a marginal agent's indifference condition to

$$\int_{0}^{1}\left[\int_{0}^{\infty}(s_k[z]-1)\,g\left(z;\hat{x}_k^{\sigma},M^{\sigma}(\theta)\right)dz\right]\frac{dm_k^{\sigma}(\theta)}{dM^{\sigma}(\theta)}dM^{\sigma}(\theta)=O(\sigma).$$

While the fundamental uncertainty (about θ) almost vanishes, the strategic uncertainty (regarding other agents' participation) remains substantial. Since $M^{\sigma}(\theta)$ is extremely sensitive to θ around the cutoff,[20] the marginal type-k agent can be very uncertain about others' participation. The strategic uncertainty faced by a marginal type-k agent is captured by $\frac{dm_k^{\sigma}(\theta)}{dM^{\sigma}(\theta)}$, which is a function of the mass of participating agents, $M^{\sigma}(\theta)$, the mass of each type j, Q_j, and the relative distance between type-j and type-k agents' switching cutoffs, $\frac{\hat{x}_j^{\sigma}-\hat{x}_k^{\sigma}}{\sigma}$. Formally, we define a function $f(\cdot)$ as follows to characterize $\frac{dm_k^{\sigma}(\theta)}{dM^{\sigma}(\theta)}$.

Definition 1. For any type $k\in\{1,2,\ldots,n\}$ and any $\{Q_j,\Delta_{k,j}\}_{j=1}^{n}$,

$$f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n}\right)\equiv\frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^{n}Q_j\phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right)}, \tag{11}$$

where m_k is a function of M implicitly defined by

$$M=\sum_{j=1}^{n}Q_j-\sum_{j=1}^{n}Q_j\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right).$$

As $\sigma\to 0$, each switching cutoff $\hat{x}_j^{\sigma}$ converges to a real number $\hat{x}_j$, and each relative distance $(\hat{x}_j^{\sigma}-\hat{x}_k^{\sigma})/\sigma$ converges to a real number or goes to infinity, denoted by $\Delta_{k,j}\in\mathbb{R}\cup\{+\infty,-\infty\}$. For a given set of masses $\{Q_j\}_{j=1}^{n}$ and relative distances $\{\Delta_{k,j}\}_{k,j=1}^{n}$, to simplify the notation, we define for each type k, $f_k(M)\equiv f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n}\right)$. Throughout the paper, $f_k(\cdot)$ has been referred to as type-k agents' *perception of participation*. Then the marginal agent's indifference condition

[20] From a marginal type-k agent's perspective, θ is highly possible to be in an $O(\sigma)$-neighborhood of $\hat{x}_k^{\sigma}$, where $m_k^{\sigma}(\theta)$, the probability that other type-k agents participate, ranges from almost 0 to almost 1, making $M^{\sigma}(\theta)$, the mass of participating agents, vary by almost Q_k in the $O(\sigma)$-neighborhood of $\hat{x}_k^{\sigma}$.

becomes

$$\int_0^\infty s_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k,M\right)f_k\left(M\right)dM\right]dz=1. \tag{12}$$

Proposition 3 characterizes the unique equilibrium following a security offering at vanishing noise.

Proposition 3. *Given a security bundle $\{(S_k,Q_k)\}_{k=1}^n$, for each $k\in\{1,2,...,n\}$, the type-k agents' switching cutoff $\hat{x}_k^\sigma$ converges to an $\hat{x}_k\in\mathbb{R}$ as $\sigma\to 0$. In particular, $\{\hat{x}_k\}_{k=1}^n$ and $\{\Delta_{k,j}\}_{j,k\in\{1,2,...,n\}}$ satisfy the equation system consisting of equation (11), equation (12),*

$$\Delta_{k-1,k}\begin{cases}=+\infty, & \text{if } \hat{x}_k>\hat{x}_{k-1}\\ =-\infty, & \text{if } \hat{x}_k<\hat{x}_{k-1}\\ \in[-\infty,+\infty], & \text{if } \hat{x}_k=\hat{x}_{k-1},\end{cases} \tag{13}$$

and

$$-\Delta_{k,j}=\Delta_{j,k}=\sum_{i=j+1}^{k}\Delta_{i-1,i}, \tag{14}$$

for each $j,k\in\{1,2,\ldots,n\}$. Conversely, if $\{\hat{x}_k\}_{k=1}^n$ and $\{\Delta_{k,j}\}_{j,k\in\{1,2,...,n\}}$ satisfy this equation system, then the type-k agents' cutoff $\hat{x}_k^\sigma$ converges to $\hat{x}_k$ as $\sigma\to 0$.

In the Appendix Section A.1, we provide intuition behind this equilibrium characterization. Notably, these conditions are not only necessary but also sufficient for $\{\hat{x}_k\}_{k=1}^n$ to be the equilibrium cutoffs. Sufficiency is important in a design problem, because it guarantees that the security bundle derived based on these conditions indeed induces the desirable outcome in equilibrium. Without loss of generality, in the rest of the paper, we number the types of securities such that $\Delta_{k-1,k}$ is nonnegative for all k.

As σ converges to 0, $m_k^\sigma(\theta)$ converges to $1\{\theta>\hat{x}_k\}$. That means, in the limit case, type-k agents almost all participate if $\theta>\hat{x}_k$ and almost all reject the offer if $\theta<\hat{x}_k$. Then we obtain the following proposition regarding the principal's expected payoff.

Proposition 4. *As $\sigma\to 0$, the mass of participating agents in state θ converges in probability to $\sum_{k=1}^n Q_k\cdot 1\{\theta>\hat{x}_k\}$, and the principal's expected payoff converges to*

$$\sum_{k=0}^{n}\int_{\hat{x}_k}^{\hat{x}_{k+1}}\left(E\left[z\mid\theta,\sum_{j=1}^{k}Q_j\right]-E\left[\sum_{j=1}^{k}S_j[z]\middle|\theta,\sum_{j=1}^{k}Q_j\right]\right)h(\theta)d\theta, \tag{15}$$

where $\hat{x}_0$ and $\hat{x}_{n+1}$ are defined as $-\infty$ and $+\infty$, respectively.

Proposition 4 implies that the mass of participating agents is essentially a step function of the state θ, which we refer to as a participation scheme in Section 3.5.

4.2 The properties of perception of participation

In this subsection, we take a closer look at the perception of participation. For notational convenience, let $\Delta_{0,1} = \Delta_{n,n+1} = +\infty$ and $\hat{x}_0 = -\hat{x}_{n+1} = -\infty$ for any n-type bundle. Define $L(k) \equiv \max\{j : \Delta_{k,j} = -\infty\}$ and $U(k) \equiv \max\{j : \Delta_{k,j} < +\infty\}$. An immediate implication of this definition is that $\Delta_{i,k}$ is finite for any $L(k) < i \leq U(k)$, so $L(i) = L(k)$ and $U(i) = U(k)$.

Proposition 5. *The perception of participation has the following properties:*

1. *For any $M \in \left(0, \sum_{j=1}^{n} Q_j\right)$, $\sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j \Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right) = M$. In particular, $\int_0^{\sum_{j=1}^{n} Q_j} f_k(y)\,dy = 1$.*

2. *The sum $\sum_{k=1}^{n} Q_k f_k(M) = 1$ for $M \in (0, \sum_{k=1}^{n} Q_k)$.*

3. *The function $f_k(M)$ is positive for $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ and 0 elsewhere.*

4. *For $L(k) < i \leq U(k)$ and $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, $\int_0^M f\left(y; \{Q_j, \Delta_{k,j}\}_{j=1}^{n}\right) dy$ is strictly increasing in $\Delta_{k,i}$.*

5. *For any $L(k) < i \leq U(k)$, if $\Delta_{k,i} = 0$, then $f_k(M) = f_i(M)$; if $\Delta_{k,i} > (<) 0$, then $f_k(M)/f_i(M)$ is strictly decreasing (increasing) in M over $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$.*

The first property follows from $\int_0^M f_k(M^\sigma(\theta))\,dM^\sigma(\theta) = m_k^\sigma(\theta) + O(\sigma)$ and confirms that $f_k(y)$ is indeed a probability distribution.

The second property states that the aggregate perception of participation equals one everywhere, which nests equation (3) in our illustrative example as a special case. This property is an immediate implication of the Bayes' rule when the agents' private information completely dominates their prior information of the state, so they basically infer the state based on their private information. This property will serve as an important constraint on security design.

The third property characterizes the support of perception of participation. For $i \leq L(k)$, the marginal type-k agents perceive the type-i agents participating almost surely; for $i > U(k)$ the former perceive the latter not participating almost surely. For other types of agents, the marginal type-k agents are uncertain whether they participate. Therefore, from the marginal type-k agents' perspective, the mass of participating agents must be within $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ almost surely and may take any value in the region.

The fourth property captures how the relative distance between cutoffs affects the perception of participation. When $\Delta_{k,i}$ increases, the marginal type-k agents perceive that the type-i agents are less likely to participate. As a result, they perceive scenarios with low levels of participation to be more likely, so their perception of participation shifts toward the left. This property has an important implication for security design. By altering $\Delta_{k,i}$ between the securities, the principal can adjust the marginal agents' perception of participation as well as how much they care about their security payments in scenarios with different levels of participation.

The fifth property concerns the relative perception of participation between different types. If $\Delta_{k,i} > 0$, the marginal type-i agents are more pessimistic about others' participation than the marginal type-k agents.

5 Security Design

This section studies the principal's security design problem. As discussed in Section 3.5, the principal's problem can be divided into two steps. The first step is to pick the optimal participation scheme, and the second step is to find the security bundle that minimizes the total expected payment to agents while implementing the optimal participation scheme. We characterize the optimal security bundle based on the second step, taking as given the optimal participation scheme. Suppose that the optimal is a T^*-cutoff participation scheme, $\{K_t, \theta_t\}_{t=1}^{T^*}$.

5.1 The assortative matching mechanism

Before formally establishing the results, we illustrate the assortative matching mechanism, which governs the principal's strategy to differentiate agents. Consider any security bundle $\{(S_k, Q_k)\}_{k=1}^{n}$ that implements $\{K_t, \theta_t\}_{t=1}^{T^*}$. Let $A_t \triangleq \{k \mid \hat{x}_k = \theta_t\}$ be the set of types whose limiting cutoffs are θ_t. The expected payment to the agents is

$$\sum_{t=1}^{T^*} \int_{\theta_t}^{+\infty} E\left[\sum_{k\in A_t} S_k[z] \mid \theta_t, K(\theta)\right] h(\theta) d\theta.$$

Note that the expected payment is determined by the aggregate securities of the types with the same limiting cutoff. Therefore, for each θ_t, the principal wants to minimize the aggregate security $\sum_{k\in A_t} S_k[z]$ subject to the constraints

$$\int_{K_{t-1}}^{K_t} E\left[s_k[z] \mid \theta_t, M\right] f_k(M)\, dM = 1, \forall k \in A_t.$$

By Proposition 5, $f_k(M)$ is 0 for M smaller than K_{t-1} or greater than K_t, because upon observing θ_t, an agent knows that those whose payoffs are smaller than θ_t will participate almost surely and those whose payoffs are greater than θ_t will not participate almost surely.

Another way to think about the problem is that if we can relax the constraints while holding the aggregate security $\sum_{k\in A_t} S_k[z]$ fixed, we can further shrink $\sum_{k\in A_t} S_k[z]$. To shed light on a potential way to relax the constraints, we combine all the constraints into the following aggregate one:

$$\sum_{k\in A_t} Q_k \int_0^{\infty} \{s_k[z]-s_k[0]\} \left\{ \int_{K_{t-1}}^{K_t} g\left(z;\theta_t,M\right) f_k\left(M\right) dM \right\} dz.$$

The first term in the integral is a security's payoff sensitivity when the project value increases from 0 to z, and the second term is marginal type-k agents' perceived probability of the project value, which is governed by their perception of participation. Lower perception of participation implies that the distribution concentrates more on low project values. Given the aggregate security $\sum_{k\in A_t} S_k[z]$, the total payoff sensitivity is fixed for each z:

$$\sum_{k\in A_t} Q_k \{s_k[z]-s_k[0]\} = \sum_{k\in A_t} S_k[z].$$

By Proposition 5, the total perception of participation, $\sum_{k\in A_t} Q_k f_k(M)$, is always 1 for $M \in (K_{t-1}, K_t)$, so the sum of marginal agents' perceived probability of the project value is also fixed for each z:

$$\begin{aligned}\sum_{k\in A_t} Q_k \int_{K_{t-1}}^{K_t} g\left(z;\hat{\theta},M\right) f_k\left(M\right) dM &= \int_{K_{t-1}}^{K_t} g\left(z;\hat{\theta},M\right) \sum_{k\in A_t} \left[Q_k f_k\left(M\right)\right] dM \\ &= \int_{K_{t-1}}^{K_t} g\left(z;\hat{\theta},M\right) dM.\end{aligned}$$

As illustrated in Section 2, the principal should differentiate agents with respect to both payoff sensitivity and perception of participation and induce an "*assortative matching*" between them.

5.2 Security Formats

First, we provide a simple observation regarding two types with the same perception of participation.

Lemma 5. *If a security bundle has $\Delta_{k-1,k}=0$ in equilibrium, it is equivalent for the principal to offer type-$(k-1)$ and type-k agents the same security $(S_{k-1}+S_k, Q_{k-1}+Q_k)$ while keeping other securities unchanged. Conversely, for any type k, there exist $\left(S'_k, Q'_k\right)$ and $\left(S''_k, Q''_k\right)$ such that it is equivalent for the principal to split the agents of this type into two types and offer them $\left(S'_k, Q'_k\right)$ and $\left(S''_k, Q''_k\right)$ respectively.*

If $\Delta_{k-1,k}=0$, by Propositions 3 and 5, $f_{k-1}(M) = f_k(M)$. Therefore, marginal type-$(k-1)$ and type-k agents evaluate their securities in the exactly same way. Essentially, they act like the same type, so the principal can directly merge them into one type without changing the equilibrium. Conversely, for type-k agents, the principal can split their aggregate security into two different ones and make an offer to them. Such a split does not change the equilibrium as long as the new securities are valued the same by marginal type-k agents.

Lemma 5 implies that to derive the optimal security bundle with the fewest types, it suffices to focus on the security bundles with all $\Delta_{k-1,k}$ being positive. Based on this, Proposition 6 characterizes the formats of the optimal securities: they must constitute a tranching structure.

Proposition 6. *The optimal security bundle with the fewest types must contain a tranching structure such that 1) agents with identical perception of participation are offered an identical tranche, and 2) agents with lower perception of participation are offered more senior tranches. That is, if* $\{(S_k, Q_k)\}_{k=1}^n$ *is optimal and have the fewest types, it must satisfy* $\Delta_{k-1,k} > 0$ *and* $S_k = T_{F_k} - T_{F_{k-1}}$ *where* $0 = F_0 < F_1 < \ldots < F_n$.

Proposition 6 follows two ideas. The first idea is that the principal and the marginal agents value cash flows differently. Let $W_k^A(z) \equiv \int_0^\infty g(z; \hat{x}_k, M) f_k(M) dM$ and $W^P(z; a, b) \equiv \int_a^b g(z; \theta, K(\theta)) h(\theta) d\theta$. According to equation (12), $W_k^A(z)$ is the marginal type-k agents' posterior probability that the project value is z and each of them receives $s_k[z]$. Hence, if $s_k[z]$ increases by 1, their expected payoff increases by $W_k^A(z)$. And $W^P(z; a, b)$ is the principal's posterior probability that the project value is z conditional on $\theta \in (a, b)$. According to equation (15), if $S_k[z]$ increases by 1, the expected payment to type-k agents increases by $W^P(z; \hat{x}_k, +\infty)$. Therefore, $W_k^A(z)$ and $W^P(z; a, b)$ represent the marginal type-k agents' and the principal's pricing kernels respectively. Due to the the SMLRP of the distribution of the project value $g(z; \theta, K)$ and the distribution of the noise $\phi(\cdot)$, $W_k^A(z)$ and $W_k^P(z)$ satisfy the following properties.

Lemma 6. *For any* k, $b > \hat{x}_k$ *and* $z_1 < z_2$, $\frac{W^P(z_2; \hat{x}_k, b)}{W^P(z_1; \hat{x}_k, b)} > \frac{W_k^A(z_2)}{W_k^A(z_1)}$ *and* $\frac{W_k^A(z_2)}{W_k^A(z_1)} > \frac{W_{k-1}^A(z_2)}{W_{k-1}^A(z_1)}$.

The first implication of Lemma 6 is that the principal cares more about the security payment at high project values than marginal agents. From the principal's perspective, conditional on a positive mass of type-k agents participating, the state is at least $\hat{x}_k$ almost surely, and the mass of participating agents is at least $K(\hat{x}_k)$ almost surely. So, the project value is higher than that expected by the marginal type-k agents. Another way to understand the difference is that the principal is concerned about the total payment to all participating agents including the marginal ones as well as those observing higher signals. The security payment of the latter agents is more at high project values. The second implication of Lemma 6 is that marginal type-k agents care more about the payment at high project values than marginal type-$(k-1)$ agents. Since $\hat{x}_{k-1} \leq \hat{x}_k$ and $\Delta_{k-1,k} > 0$, the former ones have weakly more optimistic view on the fundamental and the level of participation than the latter ones. As a result, the former ones perceive high project values to be more likely than the latter ones do.

The second idea that Proposition 6 follows is that to convince agents to participate in a cost-effective way, the principal should allocate the cash flow at certain project values to the agents who value it most. On the one hand, since marginal agents value cash flows at low project values more, the principal would be better off by giving the agents cash flows at low project values in exchange for those at high project values. However, this improvement is constrained by the monotonicity of securities and the budget constraint. Therefore, the optimal security bundle should look like a senior tranche in aggregate. On the other hand, regarding the allocation of cash flows between type-k and type-$(k-1)$ agents, the principal should allocate more cash flows at low project values to type-$(k-1)$ agents and more cash flows at high values to type-k agents. Due to the same constraint mentioned above, this idea naturally implies that the optimal security bundle should constitute a tranching structure.

Two comments about Proposition 6 are in order. First, the optimality of tranching structures does not rely on vanishing noise because the principal and marginal agents value cash flows differently in the same way as above for any σ. Second, Proposition 6 implies tranching structures but not necessarily ones with multiple tranches. It does not rule out the possibility that the optimal security bundle offers a senior tranche to all agents and they have the same perception of participation.

5.3 The Number of Security Types

In this subsection, we determine the number of the types of securities in the optimal security bundle. Proposition 7 indicates that the optimal security bundle should use up all available types.

Proposition 7. *The optimal security bundle must contain N types.*

To illustrate the intuition of Proposition 7, we take $N = 2$ as an example. Suppose Proposition 7 does not hold and the optimal security bundle with the fewest types has only one type $\{(S,K)\}$ with an equilibrium cutoff $\hat{\theta}$. As implied by Proposition 6, it must be a senior tranche, so $S = T_F$. According to Lemma 5, we can split (T_F,K) into a senior tranche (T_{F_1},Q_1) and a junior tranche $(T_F - T_{F_1}, K - Q_1)$ such that the mass K of agents behave in the same way as they do in the original equilibrium: $\hat{x}_1 = \hat{x}_2 = \hat{\theta}$ and $\Delta_{1,2} = 0$.

Consider a marginal increase of $\Delta_{1,2}$ from 0. It pushes $f_1(M)$ to shift leftward and $f_2(M)$ to shift rightward, while keeping fixed their weighted sum $Q_1 f_1(M) + (K - Q_1) f_2(M)$. As a result of the shift in perception of participation, type-1 agents now demand a better security or $F_1' > F_1$ to participate in the state $\hat{\theta}$, and type-2 agents would now be willing to accept $T_{F'} - T_{F_1'}$. A general insight in the literature on security design is that a senior tranche is less sensitive to the project value. If that is true, a small positive $\Delta_{1,2}$ will hurt type-1 agents to a lesser extent than it benefits type-2 agents, so $\left\{(T_{F_1'},Q_1),(T_{F'} - T_{F_1'}, K - Q_1)\right\}$ strictly dominates $\{(T_F,K)\}$. This insight does not hold for all ranges of project values. Specifically, in the 45-degree region of the senior tranche, the junior tranche is always worth 0 and has lower payoff sensitivities. Hence, the net impact of a marginal increase of $\Delta_{1,2}$ is not straightforward in general.

However, we can always choose to carve out a smaller senior tranche. With a smaller F_1, the senior tranche has a smaller 45-degree region and is closer to a risk-free security that has zero payoff sensitivity everywhere. The assumption $G(0;\theta,K)=0$ guarantees that a senior tranche that is sufficiently close to being risk-free always exists. With that senior tranche being offered to type-1 agents, the leftward shift in $f_1(M)$ does little harm to type-1 agents at margin but the rightward shift in $f_2(M)$ still does substantial good to type-2 agents. Therefore, a marginal increase of $\Delta_{1,2}$ is desirable for the principal.

5.4 The Benefits of Differentiation

To summarize, when the principal can flexibly offer any bundle of securities, the optimal security bundle uses multiple tranches to intentionally differentiate agents with respect to perception of participation and payoff sensitivity and match them assortatively. It is desirable for the principal to even differentiate agents with the same cutoff in equilibrium through multi-tranching. The emergence of such additional differentiation hinges on the coordination frictions induced by incomplete information.[21]

However, it should be clear that our results do not imply that differentiation in agents' payoffs per se is desirable: its desirability relies on proper design of security format. To illustrate this point, we consider the problem with a kind of restriction on security formats common in practice. In many contexts, securities can be differentiated but need to follow the same format and are thus collinear. For example, when a firm grants equity to important employees, the actual payoffs can be differentiated by heterogeneous numbers of shares but they are all proportional to the firm's equity value. Similarly, in syndicate financing, an entrepreneur borrows from investors through the same loan but may offer them different upfront fees. As discussed in Section 2.4, if only collinear securities can be offered, differentiating agents is not desirable for the principal, because differentiation in this situation only weakens the assortative matching between perception of participation and payoff sensitivity. Proposition 8 confirms this insight in the formal model.

Proposition 8. *If the principal is restricted to offering a security bundle with collinear securities, the optimal security bundle offers the same security to agents with the same limiting cutoff.*

The point that differentiation is not desirable on its own helps contrast our model to the literature on unique implementation, especially Segal (2003), Winter (2004), and Halac et al. (2020). In these models, the principal designs contracts that determine the payoffs in a complete information coordination game played by the agents, which naturally admits multiple equilibria. To capture the "strategic risk" in a complete information game, these models require the principal to evaluate

[21] It can be shown that when agents can coordinate perfectly, the optimal security bundle does not achieve such additional differentiation.

a contract according to the worst equilibrium outcome of the resultant game. This combination of the equilibrium selection and the complete information setting prohibits agents from providing assurance in a mutual way,[22] so that extreme differentiation of agents' payoffs to eliminate the "strategic risk" is always preferred, irrespective of restrictions on payoff structures. In contrast, the global game approach allows fine-tuning of strategic risk perceived by agents, $f(\Delta)$, through adjusting the relative distances between cutoffs, Δ, by security design. Whether differentiation is preferred depends on how it affects the matching between *payoff sensitivity* and *perception of participation*. It turns out that the principal's flexibility in differentiating security format is crucial.

6 Applications and Further Discussion

We briefly sketch some further discussion on the model. Please see the Online Appendix for more details.

Differentiation in perception of participation. We have shown that the optimal security design induces differentiation in agents' perception of participation. A natural idea is that it might be optimal to take such differentiation to extremes: let all $\Delta_{k-1,k}$ be infinity. This corresponds to the case where there is no strategic uncertainty between any two types of agents: marginal agents think all agents of other types either accept almost surely or reject the offer almost surely. However, this idea is not correct. The tension is that increasing the face value of a senior tranche mainly increases its value at high project values, but senior tranche holders, who have low perception of participation, do not value cash flows at high project values much. Therefore, further reducing senior tranche–holders' perception of participation may require a large increase in the face value of the senior tranche.

Zero contracting premium due to miscoordination. In the baseline model, we assume that the lowest possible project value is always zero. As a result, the contracting premium due to miscoordination, which is captured by the difference between the expected security payments to agents and their opportunity costs of participation, is always positive, and thus the principal always strictly prefers a finer differentiation. When the lowest possible project value is positive, the principal may be able to use a security bundle to achieve zero contracting premium. Since such a bundle has achieved the theoretically lowest cost, the principal does not benefit from a finer differentiation. We derive the condition for such a security bundle to exist and characterize the minimum number of types required by it.

Offering Menus to Agents. The baseline setup assumes that the principal can offer an agent one security. This is without loss of generality because the principal does not benefit from offering

[22] Halac et al. (2021) allow mutual assurance among agents by introducing incomplete information on agents' payoffs.

more than one security to any agent. Suppose that the principal offers menus to agents. It is not hard to see that each agent still follows a cutoff strategy to participate. Suppose that their cutoffs are $\{\hat{x}_k\}_{k=1}^n$, and upon observing $\hat{x}_k$, type-k agents participate and choose the security $s_k[\cdot]$. If the principal offers only $s_k[\cdot]$ to type-k agents, each agents still follows the same cutoff strategy. Therefore, offering type-k agents more securities other than $s_k[\cdot]$ does not change the expected project value, but gives them more options to maximize the expected security payment that the principal pays to them. Therefore, the principal does not benefit from offering more options.

We have formulated our model as a principal motivating agents to participate in a project. There are various examples that may fit this description.

Corporate finance. A large literature on corporate finance studies debt rollover risk that derives from investors' miscoordination. Our results directly imply that to mitigate the adverse impact of miscoordination, the firm should offer different securities to investors. Some investors have payoffs less sensitive to the state of the firm, so they are not so concerned about miscoordination and become eager to invest. Their eagerness further reassures other investors so that other investors would also like to invest despite high sensitivities of their payoffs to the state of the firm. Such differentiation can be implemented by a debt-seniority structure or uneven allocation of collateral among investors.

Financial stability. Strategic risk also undermines the stability of financial systems. A companion paper, Dai et al. (2024), studies how to design bank disclosures to achieve this goal. Following similar intuition, the paper finds that revealing banks' heterogeneous vulnerabilities to systemic risk to some extent can make the whole banking system more robust but revealing banks' heterogeneous idiosyncratic shortfall of funds does not. Our framework also provides guidance for the use of capital requirements on banks. A capital requirement increases banks' resistance to adverse shocks but also restricts banks' ability to provide welfare-improving services, so regulators would prefer to economize on the use of capital requirements. Applying our results to this setting suggests that heterogeneous capital requirement on banks could be desirable. The investors who banks have stricter capital requirements are less concerned about systemic risk, and their confidence in their banks further bolsters other investors' confidence in the whole banking system.

Employee compensation. A firm's performance depends on the effort of all employees. The design of employee compensation should take into consideration employees' concern about others' effort. Different from the existing literature (Segal, 2003; Winter, 2004; Halac et al., 2020), our results imply that a simple differentiation of employees in terms of high or low rewards may not be helpful; employee compensation should be differentiated with respect to its sensitivity to the firm's performance. Roughly, salary is the least sensitive to the firm's performance, stock & option is the most sensitive, and bonus is in the middle. The differentiation with respect to sensitivity can be implemented by offering different combinations of these parts to employees.

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Appendix

Proof of Proposition 1

First, we show that the optimal contract must result in the two banks having a common cutoff in the limit case of $\sigma \to 0$. Consider any contract $\{(c_1, d_1), (c_2, d_2)\}$ that results in two different cutoffs $\hat{x}_1$ and $\hat{x}_2$, where $\hat{x}_1 < \hat{x}_2$. Then

$$c_1 + (d_1 - c_1) P_0(\hat{x}_1) = 1$$
$$c_2 + (d_2 - c_2) P_1(\hat{x}_2) = 1.$$

Note that for $\theta \in (\hat{x}_1, \hat{x}_2)$, only bank 1 participates, and the success probability is $P_0(\theta)$. Hence, the firm would prefer bank 1 not to participate in these cases. Consider the alternative contract $\{(\rho c_1, \rho d_1), (c_2, d_2)\}$ such that

$$\rho c_1 + (\rho d_1 - \rho c_1) P_0(\hat{x}_2) = 1$$
$$c_2 + (d_2 - c_2) P_1(\hat{x}_2) = 1.$$

Since $P_0(\hat{x}_1) < P_0(\hat{x}_2)$, $\rho < 1$. Note that for all θ, the expected project value is the same under the two contracts, but the alternative contract has the lower payment to banks for $\theta \in (\hat{x}_1, \hat{x}_2)$ and the same payment in other cases. Therefore, the firm is better off by offering this contract.

Consider any contract $\{(c_1, d_1), (c_2, d_2)\}$ that results in a common cutoff $\hat{x}$ as $\sigma \to 0$. Then there exists $\Delta > 0$ such that

$$c_1 + (d_1 - c_1) p(\hat{x}, \Delta) = 1$$
$$c_2 + (d_2 - c_2) p(\hat{x}, -\Delta) = 1$$

where

$$p(\hat{x}, \pm\Delta) \triangleq P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(\pm\Delta).$$

Suppose the project value is V upon success. The firm's expected payoff is

$$\int_{-\infty}^{\hat{x}} \left[P_0(\theta) V + (1 - P_0(\theta)) \overline{C}\right] h(\theta)\, d\theta + \int_{\hat{x}}^{+\infty} \left[P_1(\theta)(V - d_1 - d_2) + (1 - P_1(\theta))\left(\overline{C} - c_1 - c_2\right)\right] h(\theta)\, d\theta.$$

First, the optimal contract must have $c_1 + c_2 = \overline{C}$. If not, consider the alternative contract $\left\{\left(\overline{C} - c_2, d_1 - \left(\overline{C} - c_1 - c_2\right)\left(\frac{1}{p(\hat{x},\Delta)} - 1\right)\right), (c_2, d_2)\right\}$. Under this contract, the two banks have the common cutoff $\hat{x}$ and the same relative distance Δ. The firm's expected payoff is

$$\begin{aligned}
&\int_{-\infty}^{\hat{x}} \left[P_0(\theta) V + (1 - P_0(\theta)) \overline{C}\right] h(\theta)\, d\theta \\
&+ \int_{\hat{x}}^{+\infty} P_1(\theta)\left[V - d_1 - d_2 + \left(\overline{C} - c_1 - c_2\right)\left(\frac{1}{p(\hat{x},\Delta)} - 1\right)\right] h(\theta)\, d\theta \\
=&\int_{-\infty}^{\hat{x}} \left[P_0(\theta) V + (1 - P_0(\theta)) \overline{C}\right] h(\theta)\, d\theta \\
&+ \int_{\hat{x}}^{+\infty} \left[P_1(\theta)(V - d_1 - d_2) + \left(\overline{C} - c_1 - c_2\right)\left(\frac{P_1(\theta)}{p(\hat{x},\Delta)} - P_1(\theta)\right)\right] h(\theta)\, d\theta \\
>&\int_{-\infty}^{\hat{x}} \left[P_0(\theta) V + (1 - P_0(\theta)) \overline{C}\right] h(\theta)\, d\theta \\
&+ \int_{\hat{x}}^{+\infty} \left[P_1(\theta)(V - d_1 - d_2) + (1 - P_1(\theta))\left(\overline{C} - c_1 - c_2\right)\right] h(\theta)\, d\theta.
\end{aligned}$$

The last inequality follows $p(\hat{x},\Delta) < \frac{P_0(\hat{x}) + P_1(\hat{x})}{2}$ and thus $\frac{P_1(\theta)}{p(\hat{x},\Delta)} > 1$ for $\theta \geq \hat{x}$.

Next, we proceed with $c_1 + c_2 = \overline{C}$. Then the goal is to minimize $d_1 + d_2$, which is

$$d_1 + d_2 = \frac{1 - c_1}{p(\hat{x},\Delta)} + \frac{1 - \overline{C} + c_1}{p(\hat{x},-\Delta)} + \overline{C}.$$

Since $p(\hat{x},\Delta) < p(\hat{x},-\Delta)$, $d_1 + d_2$ is decreasing in c_1. Hence, the optimal contract must have $c_1 = \overline{C}$, $c_2 = 0$, and

$$\begin{aligned}
d_1 + d_2 &= \frac{1 - \overline{C}}{p(\hat{x},\Delta)} + \frac{1}{p(\hat{x},-\Delta)} + \overline{C} \\
&= \frac{1 - \overline{C}}{p(\hat{x},\Delta)} + \frac{1}{P_0(\hat{x}) + P_1(\hat{x}) - p(\hat{x},\Delta)} + \overline{C}.
\end{aligned}$$

The last equation follows

$$
\begin{aligned}
p(\hat{x},\Delta)+p(\hat{x},-\Delta) &= 2P_0(\hat{x})+[P_1(\hat{x})-P_0(\hat{x})]\cdot[f(\Delta)+f(-\Delta)] \\
&= P_0(\hat{x})+P_1(\hat{x}).
\end{aligned}
$$

The derivative of d_1+d_2 with respect to $p(\hat{x},\Delta)$ is

$$
-\frac{1-\overline{C}}{p(\hat{x},\Delta)^2}+\frac{1}{[P_0(\hat{x})+P_1(\hat{x})-p(\hat{x},\Delta)]^2},
$$

so d_1+d_2 attains the minimum at

$$
p(\hat{x},\Delta)=\frac{P_0(\hat{x})+P_1(\hat{x})}{1+\frac{1}{\sqrt{1-\overline{C}}}}.
$$

Since $p(\hat{x},\Delta)\in[P_0(\hat{x}),P_1(\hat{x})]$, $p(\hat{x},\Delta)=\frac{P_0(\hat{x})+P_1(\hat{x})}{1+\frac{1}{\sqrt{1-\overline{C}}}}$ is attainable if $\frac{P_0(\hat{x})}{P_1(\hat{x})}\leq\sqrt{1-\overline{C}}$. Otherwise, $p(\hat{x},\Delta)=P_0(\hat{x})$ attains the minimum.

Taken together, if $\frac{P_0(\hat{x})}{P_1(\hat{x})}\leq\sqrt{1-\overline{C}}$, the optimal contract is $\left\{\left(\overline{C},\frac{1-\overline{C}+\sqrt{1-\overline{C}}}{P_0(\hat{x})+P_1(\hat{x})}+\overline{C}\right),\left(0,\frac{1+\sqrt{1-\overline{C}}}{P_0(\hat{x})+P_1(\hat{x})}\right)\right\}$; if $\frac{P_0(\hat{x})}{P_1(\hat{x})}>\sqrt{1-\overline{C}}$, the optimal contract is $\left\{\left(\overline{C},\frac{1-\overline{C}}{P_0(\hat{x})}+\overline{C}\right),\left(0,\frac{1}{P_1(\hat{x})}\right)\right\}$.

Proof of Proposition 2

Following the proof of Proposition 1, the optimal contract must result in the two banks having a common cutoff in the limit case of $\sigma\to 0$. Consider two collinear but not identical securities $\{(c_1,d_1),(\rho c_1,\rho d_1)\}$ with $\rho<1$ that result in a common cutoff $\hat{x}$ as $\sigma\to 0$. Then there exists $\Delta>0$ such that

$$
\begin{aligned}
c_1+(d_1-c_1)p(\hat{x},\Delta) &= 1, \\
c_1+(d_1-c_1)p(\hat{x},-\Delta) &= \frac{1}{\rho}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
1+\frac{1}{\rho} &= 2c_1+(d_1-c_1)[p(\hat{x},\Delta)+p(\hat{x},-\Delta)]. \\
&= 2c_1+(d_1-c_1)[P_0(\hat{x})+P_1(\hat{x})]
\end{aligned}
$$

Consider alternative identical securities $\left\{\left(\frac{2}{1+\frac{1}{\rho}}c_1, \frac{2}{1+\frac{1}{\rho}}d_1\right), \left(\frac{2}{1+\frac{1}{\rho}}c_1, \frac{2}{1+\frac{1}{\rho}}d_1\right)\right\}$. Since

$$\begin{aligned}
&\frac{2}{1+\frac{1}{\rho}}c_1 + \left(\frac{2}{1+\frac{1}{\rho}}d_1 - \frac{2}{1+\frac{1}{\rho}}c_1\right) \cdot p(\hat{x},0) \\
=&\frac{1}{1+\frac{1}{\rho}}\left\{2c_1 + (d_1 - c_1)\left[P_0(\hat{x}) + P_1(\hat{x})\right]\right\} = 1,
\end{aligned}$$

the two securities also result in a common cutoff $\hat{x}$ as $\sigma \to 0$. However, their aggregate payment is smaller than that of the original two securities, by Cauchy-Schwarz Inequality:

$$\frac{2}{1+\frac{1}{\rho}} \times 2 < 1 + \rho.$$

Proof of Proposition 5

Property 1. Suppose for $M \in \left(0, \sum_{j=1}^n Q_j\right)$, $\gamma_k(M)$ solves

$$M = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,j}\right).$$

Note that $\gamma_k(0) = 0$. Property 1 holds because

$$\begin{aligned}
\int_0^M f_k(y)\,dy =& \int_{m_k=\gamma_k(0)}^{\gamma_k(M)} \frac{d(1-m_k)/d\Phi^{-1}(1-m_k)}{-d\left[\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)\right]/d\Phi^{-1}(1-m_k)} \\
&\times d\left[\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)\right] \\
=& \int_{m_k=\gamma_k(M)}^{\gamma_k(0)} d(1-m_k) = \gamma_k(M).
\end{aligned}$$

Property 2. Since $\gamma_k(M)$ is unique for any k and $M \in \left(0, \sum_{j=1}^n Q_j\right)$, then $\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,j} = \Phi^{-1}(1-\gamma_j(M))$. Therefore,

$$M = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-\gamma_j(M))\right) = \sum_{j=1}^n Q_j \gamma_j(M) = \int_0^M \sum_{j=1}^n Q_j f_j(y)\,dy,$$

Taking derivative with respect to M, we obtain $\sum_{j=1}^n Q_j f_j(M) = 1$.

Property 3. Note that for $j \le L(k)$, $\Delta_{k,j} = -\infty$. According to Property 1, if $\int_0^M f_k(y)\,dy > 0$, then for $j \le L(k)$,

$$\Phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,j}\right)=\Phi(-\infty)=0,$$

and

$$\Phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,k}\right)=1-\int_0^M f_k(y)\,dy<1,$$

so $M > \sum_{j=1}^{L(k)} Q_j$. If $\int_0^M f_k(y)\,dy = 0$, then for $j > L(k)$,

$$\Phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,j}\right)=\Phi(+\infty)=1,$$

so $M \le \sum_{j=1}^{L(k)} Q_j$. Combing the two arguments, we obtain $\int_0^M f_k(y)\,dy > 0 \Leftrightarrow M > \sum_{j=1}^{L(k)} Q_j$. Likewise, we obtain $\int_0^M f_k(y)\,dy < 1 \Leftrightarrow M < \sum_{j=1}^{U(k)} Q_j$.

For $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, $\int_0^M f_k(y)\,dy \in (0,1)$, Property 1 implies

$$\sum_{j=1}^{U(k)} Q_j - \sum_{j=L(k)+1}^{U(k)} Q_j \Phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,j}\right)=M.$$

Taking derivative with respect to M,

$$f_k(M)=\frac{\phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)\right)}{\sum_{j=L(k)+1}^{U(k)} Q_j \phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,j}\right)},$$

which is positive because $\Delta_{k,j}$ here are all finite.

Property 4. For $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$,

$$\sum_{j=1}^{U(k)} Q_j - \sum_{j=L(k)+1}^{U(k)} Q_j \Phi\left(\Phi^{-1}\left(1-\int_0^M f_k(y)\,dy\right)+\Delta_{k,j}\right)=M.$$

It is straightforward to see the left-hand side is strictly increasing in $\int_0^M f_k(y)\,dy$ and strictly decreasing in $\Delta_{k,i}$. So, $\int_0^M f_k(y)\,dy$ is strictly increasing in $\Delta_{k,i}$.

Property 5. Consider $L(k) < i \le U(k)$. Then $L(i) = L(k)$ and $U(i) = U(k)$. Therefore, $f_k(M)$ and $f_i(M)$ are both positive for $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, so their ratio is well defined in the region.

By $\Phi^{-1}(1-\gamma_k(M))+\Delta_{k,i}=\Phi^{-1}(1-\gamma_i(M))$,

$$d\gamma_i(M)=\frac{\phi\left(\Phi^{-1}(1-\gamma_k(M))+\Delta_{k,i}\right)}{\phi\left(\Phi^{-1}(1-\gamma_k(M))\right)}d\gamma_k(M),$$

so

$$\frac{f_k(M)}{f_i(M)}=\frac{d\gamma_k(M)/dM}{d\gamma_i(M)/dM}=\frac{\phi\left(\Phi^{-1}(1-\gamma_k(M))\right)}{\phi\left(\Phi^{-1}(1-\gamma_k(M))+\Delta_{k,i}\right)}.$$

By SMLRP, we obtain Property 5.

Proof of Proposition 6

In this proof, we always take $\{(\hat{x}_k,Q_k,\Delta_{k-1,k})\}_{k=1}^n$ as given and only alter the securities $\{S_k\}_{k=1}^n$. Hence, all agents' perception of participation stays unchanged.

First, consider the first two types. The expected payment to type-1 and type-2 agents is

$$\int_{\hat{x}_1}^{+\infty}E\left[S_1[z]\mid\theta,K(\theta)\right]h(\theta)d\theta+\int_{\hat{x}_2}^{+\infty}E\left[S_2[z]\mid\theta,K(\theta)\right]h(\theta)d\theta.$$

Let $S_1'[z]\equiv\min\{S_1[z]+S_2[z],F'\}$ and $S_2'[z]\equiv S_1[z]+S_2[z]-S_1'[z]$, where F' is the minimum value such that $\int_0^\infty S_1'[z]W_1^A(z)dz=Q_1$.

I show $\int_0^\infty S_2'[z]W_2^A(z)dz\geq\int_0^\infty S_2[z]W_2^A(z)dz$. Since $\int_0^\infty S_1[z]W_1^A(z)dz=Q_1$, there must exists a minimum $\tilde{z}\geq 0$ such that $S_1[z]-S_1'[z]$ is nonpositive for $z\leq\tilde{z}$ and nonnegative for $z>\tilde{z}$. So, $\int_0^{\tilde{z}}\left(S_1'[z]-S_1[z]\right)W_1^A(z)dz=\int_{\tilde{z}}^{+\infty}\left(S_1[z]-S_1'[z]\right)W_1^A(z)dz$. Then

$$\begin{aligned}\int_0^\infty S_2'[z]W_2^A(z)dz-\int_0^\infty S_2[z]W_2^A(z)dz&=\int_0^\infty\left(S_1[z]-S_1'[z]\right)W_2^A(z)dz\\=\int_0^{\tilde{z}}\left(S_1[z]-S_1'[z]\right)\frac{W_2^A(z)}{W_1^A(z)}W_1^A(z)dz&+\int_{\tilde{z}}^{+\infty}\left(S_1[z]-S_1'[z]\right)\frac{W_2^A(z)}{W_1^A(z)}W_1^A(z)dz.\end{aligned}$$

Since $W_2^A(z)/W_1^A(z)$ is strictly increasing,

$$\begin{aligned}&\int_0^\infty S_2'[z]W_2^A(z)dz-\int_0^\infty S_2[z]W_2^A(z)dz\\&\geq\int_0^{\tilde{z}}\left(S_1[z]-S_1'[z]\right)\frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})}W_1^A(z)dz+\int_{\tilde{z}}^{+\infty}\left(S_1[z]-S_1'[z]\right)\frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})}W_1^A(z)dz\\&=\frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})}\int_0^{+\infty}\left(S_1[z]-S_1'[z]\right)W_1^A(z)dz=0\end{aligned}$$

Next, I show

$$\int_{\hat{x}_1}^{\hat{x}_2}\left[\int_0^{+\infty}S_1'[z]g\left(z;\theta,K(\theta)\right)dz\right]h(\theta)d\theta\leq\int_{\hat{x}_1}^{\hat{x}_2}\left[\int_0^{+\infty}S_1[z]g\left(z;\theta,K(\theta)\right)dz\right]h(\theta)d\theta.$$

If $\hat{x}_1 = \hat{x}_2$, it is obvious. If $\hat{x}_1 < \hat{x}_2$, the inequality is equivalent to

$$\int_0^{+\infty} S_1'[z] W^P(z;\hat{x}_1,\hat{x}_2) dz \leq \int_0^{+\infty} S_1[z] W^P(z;\hat{x}_1,\hat{x}_2) dz.$$

Since $W^P(z;\hat{x}_1,\hat{x}_2)/W_1^A(z)$ is strictly increasing, we prove it following the same logic as above.

Let

$$\rho_2 = \frac{\int_0^\infty S_2'[z] W_2^A(z) dz}{\int_0^\infty S_2[z] W_2^A(z) dz}$$

such that $\int_0^\infty \rho_2 S_2'[z] W_2^A(z) dz = Q_2$. Then $\rho_2 \leq 1$. So, the principal can replace S_1 and S_2 with S_1' and $\rho_2 S_2'$ to implement the participation scheme. Offering them, the expected payment to type-1 and type-2 agents is lower, i.e.,

$$\begin{aligned}
&\int_{\hat{x}_1}^{\hat{x}_2} \left[\int_0^{+\infty} S_1'[z] g(z;\theta,K(\theta)) dz \right] h(\theta) d\theta + \int_{\hat{x}_2}^{+\infty} \left[\int_0^{+\infty} \left(S_1' + \rho_2 S_2'\right)[z] g(z;\theta,K(\theta)) dz \right] h(\theta) d\theta \\
\leq &\int_{\hat{x}_1}^{\hat{x}_2} \left[\int_0^{+\infty} S_1[z] g(z;\theta,K(\theta)) dz \right] h(\theta) d\theta + \int_{\hat{x}_2}^{+\infty} \left[\int_0^{+\infty} (S_1 + S_2)[z] g(z;\theta,K(\theta)) dz \right] h(\theta) d\theta.
\end{aligned}$$

Since the expected payment to other types of agents does not change, the total expected payment is lower. The equality holds only when $S_1[z] = S_1'[z]$.

Second, consider S_1' and S_3. Likewise, we construct $S_1''[z] \equiv \min\{S_1'[z] + S_3[z], F''\}$, $S_3'[z] \equiv S_1'[z] + S_3[z] - S_1''[z]$, and ρ_3. Following the above analysis, we can show that the total expected payment is strictly lower if the principal offers S_1'' and $\rho_3 S_3$ instead of S_1' and S_3. If $F'' > F'$, then $S_1''[z] \geq S_1'[z]$ for a positive measure of z, so $\int_0^\infty S_1''[z] W_1^A(z) dz > \int_0^\infty S_1'[z] W_1^A(z) dz = Q_1$. Contradiction! So, $F'' \leq F'$ and $S_1''[z] = \min\{\sum_{k=1}^3 S_k[z], F''\}$. Iterating this procedure with all remaining contracts, we end up with $S_1'''[z] \equiv \min\{\sum_{k=1}^n S_k[z], F'''\}$ and a lower total expected payment.

Third, consider T_{F_1}, where F_1 is the minimum value such that $\int_0^\infty T_{F_1}[z] W_1^A(z) dz = Q_1$. Following the above analysis, we know $F_1 \leq F'''$, so it is feasible to the offer T_{F_1} to type-1 agents without changing the contracts to other types. The total expected payment is lower if the principal offers T_{F_1} to type-1 agents instead of S_1'''.

Fourth, given that the first k contracts take the form $T_{F_k} - T_{F_{k-1}}$ where $0 = F_0 < F_1 < \ldots < F_k$, iterating the above three steps on the $(k+1)$th contract, we can show that the total expected payment is lower if it takes the form $T_{F_{k+1}} - T_{F_k}$ where $F_{k+1} > F_k$. Finally, we end up with a tranching structure. Moreover, from the above proof, we can see that if such a tranching structure has a strictly lower total expected payment than any other security bundle does.

Proof of Proposition 7

Part I: constructing an alternative security bundle

Suppose that Proposition 7 does not hold. That means, there exists an optimal security bundle with n types $(n < N)$. We pick the **first** one of them and split it into two types in the way described in Lemma 5 such that all agents use the same strategy as before. Note that this $(n+1)$-type security bundle should also be the optimal. Denote it as $\{(S_j, Q_j)\}_{j=1}^{n+1}$ and its resultant equilibrium as $\{(\hat{x}_j, \Delta_{j-1,j})\}_{j=1}^{n+1}$. In the rest of the proof, $\{(S_j, Q_j)\}_{j=1}^{n+1}$ is referred to as the original security bundle. We intend to show that the original security bundle cannot be the optimal. To this end, it is without loss of generality to assume that this security bundle contains a tranching structure as in Proposition 6. So, $S_j = T_{F_j} - T_{F_{j-1}}$ where $0 = F_0 < F_1 < \ldots < F_{n+1}$.

Since the first type and the second type are the two created by the split, $\hat{x}_1 = \hat{x}_2$, $\Delta_{1,2} = 0$, $L(1) = L(2) = 0$, and $U(1) = U(2)$. Keeping all $\hat{x}_j$ unchanged, consider an alternative equilibrium $\{(\hat{x}_j, \Delta'_{j-1,j})\}_{j=1}^{n+1}$ where $\eta \in [0, +\infty)$ and

$$\Delta'_{j-1,j} = \begin{cases} +\infty, & \text{if } j = 1 \\ \Delta_{1,2} + \eta, & \text{if } j = 2 \\ \Delta_{j-1,j}, & \text{if } j \geq 3 \end{cases}.$$

It is easy to see that this equilibrium satisfies equation (11). Then $\forall k \neq 1$,

$$\Delta'_{k,j} = \begin{cases} \Delta_{k,j}, & \text{if } j \neq 1, \\ \Delta_{k,1} - \eta, & \text{if } j = 1, \end{cases} \text{ and } \Delta'_{1,j} = \begin{cases} \Delta_{1,j} + \eta, & \text{if } j \neq 1, \\ 0, & \text{if } j = 1, \end{cases}$$

Since η is finite, $L(j)$ and $U(j)$ remain the same for all j in the alternative equilibrium. Keeping all Q_j unchanged, consider an alternative security bundle $\{(S'_j, Q_j)\}_{j=1}^{n+1}$.

$$S'_j = \begin{cases} T_{F'_1}, & \text{if } j = 1 \\ T_{F'_2} - T_{F'_1}, & \text{if } j = 2 \\ \rho_j S_j, & \text{if } j \geq 3 \end{cases}.$$

We claim that if $\{(S'_j, Q_j)\}_{j=1}^{n+1}$ results in the equilibrium $\{(\hat{x}_j, \Delta'_{j-1,j})\}_{j=1}^{n+1}$, all ρ_j must be weakly smaller than 1. Since $\{(S_j, Q_j)\}_{j=1}^{n+1}$ results in $\{(\hat{x}_j, \Delta_{j-1,j})\}_{j=1}^{n+1}$, by the indifference condition of the marginal type-k agents, equation (12), we obtain

$$\int_0^\infty S_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k,M\right) f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right) dM\right] dz = Q_k$$
$$\Leftrightarrow \int_0^\infty S_k[z]\left\{\int_{M=\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_k,M\right) d\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right) dy\right]\right\} dz = Q_k$$
$$\Leftrightarrow \int_0^\infty S_k[z]\left\{g\left(z;\hat{x}_k,\sum_{j=1}^{U(k)} Q_j\right) - \int_{M=\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right) dy\right] dg\left(z;\hat{x}_k,M\right)\right\} dz = Q_k.$$

Consider $\{(S'_j,Q_j)\}_{j=1}^{n+1}$ that results in $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$. For $k \geq 3$, the indifference condition of the marginal type-k agents implies

$$\rho_k \int_0^\infty S_k[z]\left\{g\left(z;\hat{x}_k,\sum_{j=1}^{U(k)} Q_j\right) - \int_{M=\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta'_{k,j}\}_{j=1}^{n+1}\right) dy\right] dg\left(z;\hat{x}_k,M\right)\right\} dz = Q_k.$$

Since $\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right) dy$ is weakly increasing in $\Delta_{k,j}$ and

$$\Delta'_{k,j} = \begin{cases} \Delta_{k,j}, & \text{if } j \neq 1 \\ \Delta_{k,1} - \eta, & \text{if } j = 1 \end{cases},$$

we obtain

$$\int_{M=\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta'_{k,j}\}_{j=1}^{n+1}\right) dy\right] dg\left(z;\hat{x}_k,M\right)$$
$$\leq \int_{M=\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right) dy\right] dg\left(z;\hat{x}_k,M\right).$$

So $\rho_k \leq 1$.

If there exists a bundle that can result in the equilibrium $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$ with F'_2 strictly smaller than F_2, then this bundle can implement the participation scheme at a strictly lower cost than the original optimal one. In the second part of the proof, we focus on F'_1 and F'_2.

Part II: there exists positive Q_1 and η such that $F'_2 < F_2$.

Note that the original bundle is created by splitting the first type of the optimal contract into two. This implies that Q_1+Q_2 and all Q_k for $k \geq 3$ are fixed. So is $\sum_{j=1}^{U(1)} Q_j$. Hence, we have two choice variables, Q_1 and η, for constructing the alternative bundle. We denote the perception of participation of the first two types as $f_1(M;Q_1,\eta)$ and $f_2(M;Q_1,\eta)$ respectively. Particularly, at $\eta = 0$,

$$f_1(M;Q_1,0) = f_2(M;Q_1,0) = f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)$$

does not vary with Q_1, which follows Lemma 5.

According to the indifference condition of marginal type-1 agents, equation (12), we have

$$Q_1 = \int_0^{F_1'} (z - F_0) \left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1, M) f_1(M; Q_1, \eta) \, dM \right] \cdot dz + \int_{F_1'}^{+\infty} (F_1' - F_0) \left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1, M) f_1(M; Q_1, \eta) \, dM \right] \cdot dz. \tag{16}$$

Note that at $\eta = 0$, $F_1' = F_1$. Holding Q_1 fixed and taking derivative of the indifference condition at $\eta = 0$, we obtain

$$\frac{dF_1'}{d\eta} \left[1 - \int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1, M) f_1(M; Q_1, 0) \, dM \right] = - \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_1(y; Q_1, 0) \, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z;\hat{x}_1, M) \, dz \right].$$

Likewise, for marginal type-2 agents,

$$Q_2 = \int_{F_1'}^{F_2'} (z - F_1') \left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1, M) f_2(M; Q_1, \eta) \, dM \right] \cdot dz + \int_{F_2'}^{+\infty} (F_2' - F_1') \left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1, M) f_2(M; Q_1, \eta) \, dM \right] \cdot dz.$$

Note that at $\eta = 0$, $F_2' = F_2$, and $f_1(M; Q_1, 0) = f_2(M; Q_1, 0)$. Holding Q_2 fixed and taking derivative of the indifference condition at $\eta = 0$, we obtain

$$\begin{aligned} &\frac{dF_2'}{d\eta} \left[1 - \int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_2;\hat{x}_1, M) f_1(M; Q_1, 0) \, dM \right] \\ =&\frac{dF_1'}{d\eta} \left[1 - \int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1, M) f_1(M; Q_1, 0) \, dM \right] - \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_2(y; Q_1, 0) \, dy}{d\eta} \cdot d\left[\int_{F_1}^{F_2} G(z;\hat{x}_1, M) \, dz \right] \\ =& - \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_1(y; Q_1, 0) \, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z;\hat{x}_1, M) \, dz \right] + \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_2(y; Q_1, 0) \, dy}{d\eta} \cdot d\left[\int_{F_1}^{F_2} G(z;\hat{x}_1, M) \, dz \right]. \end{aligned}$$

To quantify $d \int_0^M f_1(y; Q_1, 0) \, dy / d\eta$, we resort to the first property in Proposition 5 and obtain

$$\sum_{j=1}^{n+1} Q_j - \sum_{j=2}^{n+1} Q_j \Phi \left(\Phi^{-1} \left(1 - \int_0^M f_1(y; Q_1, 0) \, dy \right) + \Delta_{1,j} + \eta \right) - Q_1 \left(1 - \int_0^M f_1(y; Q_1, 0) \, dy \right) = M$$

and

$$\sum_{j=1}^{n+1} Q_j - \sum_{j=2}^{n+1} Q_j \Phi \left(\Phi^{-1} \left(1 - \int_0^M f_2(y; Q_1, 0) \, dy \right) + \Delta_{2,j} \right) - Q_1 \Phi \left(\Phi^{-1} \left(1 - \int_0^M f_2(y; Q_1, 0) \, dy \right) + \Delta_{2,1} - \eta \right) = M.$$

Taking derivative with respect to η at $\eta = 0$, we have

$$\left[\sum_{j=3}^{n+1} Q_j \frac{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right)} + Q_1 + Q_2\right] \frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}$$
$$= \sum_{j=2}^{n+1} Q_j \phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right) > 0$$

and

$$\left[\sum_{j=3}^{n+1} Q_j \frac{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right)} + Q_1 + Q_2\right] \frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\frac{1}{Q_1}$$
$$= -\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right) < 0.$$

Here we use $\Delta_{1,2} = 0$ and thus $\int_0^M f_1(y;Q_1,0)\,dy = \int_0^M f_2(y;Q_1,0)\,dy$ at $\eta = 0$.

We claim that there exists Q_1 such that $dF_2'/d\eta < 0$. According to equation (16), it is easy to see that at $\eta = 0$, F_1 goes to 0 as Q_1 goes to 0, and

$$dQ_1 = dF_1\left[1-\int_0^{\sum_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\, f_1(M;Q_1,0)\,dM\right].$$

By L'Hospital's rule,

$$\lim_{Q_1\to 0}\frac{F_1}{Q_1} = \lim_{Q_1\to 0}\frac{dF_1}{dQ_1} = \lim_{F_1\to 0}\frac{1}{1-\int_0^{\sum_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\, f_1(M;Q_1,0)\,dM} = 1.$$

Since $\int_0^M f_1(y;Q_1,0)\,dy$ does not vary with Q_1, so is $\frac{d\int_0^M f_2(y;Q_1,0)dy}{d\eta}\frac{1}{Q_1}$. Since $G(z;\hat{x}_1,M)$ is decreasing in M for any z,

$$\lim_{Q_1\to 0}\frac{1}{Q_1}\int_{M=0}^{\sum_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_{F_1}^{F_2} G(z;\hat{x}_1,M)\,dz\right]$$
$$= \int_{M=0}^{\sum_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\frac{1}{Q_1}\cdot d\left[\int_0^{F_2} G(z;\hat{x}_1,M)\,dz\right] > 0.$$

We only need to show that

$$\lim_{Q_1\to 0}\frac{1}{Q_1}\int_{M=0}^{\sum_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1} G(z;\hat{x}_1,M)\,dz\right] = 0.$$

Consider $M(F)$ such that $G(F;\hat{x}_1,M(F)) = M(F)$. It is straightforward to see that $M(F)$ is strictly increasing in F. Since $G(0;\hat{x}_1,M)=0$ for any M, $\lim_{F\to 0} M(F)=0$. Note that

$$\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta} \le \frac{\sum_{j=2}^{n+1} Q_j\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{Q_1+Q_2}.$$

According to Lemma 3, $\phi(\cdot)$ is bounded, and

$$\lim_{M\to 0}\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right) = \lim_{\varepsilon\to+\infty}\phi(\varepsilon)=0.$$

Therefore, for any $\overline{M}$, there exists finite $\chi(\overline{M})$ such that $d\int_0^M f_1(y;Q_1,0)\,dy/d\eta < \chi(\overline{M})$ for any $M\in[0,\overline{M}]$ and $\chi(\overline{M})$ converges 0 as $\overline{M}$ goes to 0. Then

$$\begin{aligned}
&\left|\frac{1}{Q_1}\int_{M=0}^{\sum_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\right|\\
\le&\left|\frac{1}{Q_1}\int_{M=0}^{M(F_1)}\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\right|+\left|\frac{1}{Q_1}\int_{M=M(F_1)}^{\sum_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\right|\\
\le&\left|\frac{1}{Q_1}\int_{M=0}^{M(F_1)}\chi(M(F_1))\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\right|+\left|\frac{1}{Q_1}\int_{M=M(F_1)}^{\sum_{j=1}^{U(1)}Q_j}\chi\left(\sum_{j=1}^{U(1)}Q_j\right)\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\right|\\
\le&\frac{\chi(M(F_1))}{Q_1}\left|\int_0^{F_1}G(z;\hat{x}_1,0)\,dz-\int_0^{F_1}G(z;\hat{x}_1,M(F_1))\,dz\right|+\frac{\chi\left(\sum_{j=1}^{U(1)}Q_j\right)}{Q_1}\left|\int_0^{F_1}G(z;\hat{x}_1,M(F_1))\,dz-\int_0^{F_1}G\left(z;\hat{x}_1,\sum_{j=1}^{U(1)}Q_j\right)dz\right|\\
\le&\frac{\chi(M(F_1))}{Q_1}\left|\int_0^{F_1}G(z;\hat{x}_1,0)\,dz\right|+\frac{\chi\left(\sum_{j=1}^{U(1)}Q_j\right)}{Q_1}\left|\int_0^{F_1}G(z;\hat{x}_1,M(F_1))\,dz\right|\\
\le&\frac{F_1}{Q_1}\chi(M(F_1))+\frac{F_1}{Q_1}\chi\left(\sum_{j=1}^{U(1)}Q_j\right)G(F_1;\hat{x}_1,M(F_1))=\frac{F_1}{Q_1}\chi(M(F_1))+\frac{F_1}{Q_1}\chi\left(\sum_{j=1}^{U(1)}Q_j\right)M(F_1).
\end{aligned}$$

As $Q_1\to 0$, $F_1/Q_1\to 1$ and $F_1\to 0$, so $M(F_1)\to 0$. We obtain

$$\frac{1}{Q_1}\int_{M=0}^{\sum_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]\to 0.$$

We confirm that there exists Q_1 such that $dF_2'/d\eta<0$.

To sum up, we prove that there exists positive Q_1 and η such that $F_2'<F_2$. With such Q_1 and η, $\{(S_j',Q_j)\}_{j=1}^{n+1}$ implements the same participation scheme as $\{(S_j,Q_j)\}_{j=1}^{n+1}$ but its total expected payment to the agents is strictly lower. Hence, $\{(S_j,Q_j)\}_{j=1}^{n+1}$ cannot be optimal, and the optimal security bundle must have N types.

Proof of Proposition 8

Without loss of generality, suppose the firm offers $\left\{\left(\frac{Q_k}{p_k}s[z], Q_k\right)\right\}_{k=1}^{n}$ and the set of types with the limiting cutoff θ_t is A_t. Then according to Proposition 3, for $k \in A_t$,

$$\int_0^\infty s[z]\left[\int_0^\infty g\left(z;\theta_j,M\right)f_k\left(M\right)dM\right]dz = p_k.$$

Multiplying by Q_k and summing over all k in A_t,

$$\int_0^\infty s[z]\left[\int_0^\infty g\left(z;\theta_j,M\right)\sum_{k\in A_t}Q_k f_k\left(M\right)dM\right]dz = \sum_{k\in A_t}Q_k p_k.$$

According to Proposition 5, $\sum_{k\in A_t}Q_k f_k\left(M\right) = 1$ for $M \in \left(K_{j-1}, K_j\right)$, so

$$\int_0^\infty s[z]\left[\int_{K_{j-1}}^{K_j} g\left(z;\theta_j,M\right)dM\right]dz = \sum_{k\in A_t}Q_k p_k.$$

Note that the aggregate security of the types in A_t is $S_j[\cdot] = \sum_{k\in A_t}\frac{Q_k}{p_k}s[\cdot]$. Then

$$\int_0^\infty S_j[z]\left[\int_{K_{j-1}}^{K_j} g\left(z;\theta_j,M\right)dM\right]dz = \sum_{k\in A_t}\frac{Q_k}{p_k}\cdot\sum_{k=1}^{n}Q_k p_k.$$

Let

$$\delta = \frac{\left(\sum_{k\in A_t}Q_k\right)^2}{\sum_{k\in A_t}\frac{Q_k}{p_k}\cdot\sum_{k\in A_t}Q_k p_k}.$$

When p_k are not all equal for all k in A_t, by Cauchy-Schwarz Inequality, $\delta < 1$. Consider that the principal offers $\delta/\sum_{k\in A_t}Q_k\cdot S[\cdot]$ to each of these agents instead. Then

$$\int_0^\infty \delta/\sum_{k\in A_t}Q_k\cdot S[z]\left[\int_{K_{j-1}}^{K_j} g\left(z;\theta_j,M\right)\frac{1}{\sum_{k\in A_t}Q_k}dM\right]dz = 1.$$

In this case, all these agents have the same perception of participation, which is a uniform distribution over $\left(K_{j-1}, K\right)$. Hence, their indifference conditions are satisfied at θ_j. That means, this alternative security bundle can induce the same set of agents to have the same common cutoff θ_j but at a strictly lower expected cost. Proposition 8 is proved.

Online Appendix

A Additional Discussion

A.1 The Intuition behind the Equilibrium Following a Security Offering

Here, we provide the intuition behind the equilibrium characterization in Proposition 3. Take two-type bundles as an example. Let $\hat{x}_1^\sigma$ and $\hat{x}_2^\sigma$ denote the two equilibrium switching cutoffs of the type-1 and the type-2 agents, respectively. If the two securities are equally attractive, both types of agents choose the same switching cutoff $\hat{x}_1^\sigma = \hat{x}_2^\sigma$, and their respective marginal agents share the same view on θ as well as the same belief on the mass of participating agents. At vanishing σ, the same view on θ results in identical conditional p.d.f. of the project value, i.e., $g(\cdot;\hat{x}_1,M) = g(\cdot;\hat{x}_2,M)$ for any given mass of participating agents M; and the same belief on the mass of participating agents is captured by the same *perception of participation* $f_1(M) = f_2(M) = 1/(Q_1+Q_2)$, where $M \in [0, Q_1+Q_2]$. As a result, the marginal agents of both types share the same p.d.f. of the project value z, i.e., $\int_0^\infty g(z;\hat{x}_1,M)\, f_1(M)\, dM = \int_0^\infty g(z;\hat{x}_2,M)\, f_2(M)\, dM$.

Now suppose security $s_1[\cdot]$ is *slightly* more attractive than $s_2[\cdot]$, resulting in $\hat{x}_1^\sigma < \hat{x}_2^\sigma$, i.e., the type-1 agents are more eager to participate. In this case, the limit cutoffs could remain identical, i.e., $\hat{x}_1 = \hat{x}_2$, but the limit relative distance becomes strictly positive, i.e., $\Delta_{1,2} = \lim_{\sigma\to 0}(\hat{x}_2^\sigma - \hat{x}_1^\sigma)/\sigma > 0$, resembling two different channels through which the security design may affect the agents' equilibrium behavior. The first one is the fundamental channel. Since $\hat{x}_1 = \hat{x}_2$, the marginal agents of both types share almost the same belief about the fundamental state θ. Hence, in this case, the fundamental channel does not come into effect, because for any given mass of participating agents M, the marginal agents of both types hold almost the same conditional p.d.f. of the project value z, i.e., $g(\cdot;\hat{x}_1,M) = g(\cdot;\hat{x}_2,M)$. However, since $\Delta_{1,2} > 0$, they perceive others' participation differently. This is the strategic channel through which the security design take effects. In particular, the marginal type-1 agents perceive that type-2 agents are less likely to participate while the marginal type-2 agents perceive the opposite about type-1. This is captured by that the type-1 perception of participation $f_1(\cdot)$ is leftward-tilted relative to the type-2 perception of participation $f_2(\cdot)$, i.e., as probability densities of the mass of participating agents, $f_1(\cdot)$ is first order stochastically dominated by $f_2(\cdot)$. Since $g(\cdot;\theta,M)$ satisfies SMLRP, $\int_0^\infty g(z;\hat{x}_1,M)\, f_1(M)\, dM$ is also first order stochastically dominated by $\int_0^\infty g(z;\hat{x}_2,M)\, f_2(M)\, dM$, meaning that regarding the project value z, the marginal type-1 agents are more pessimistic relative to the marginal type-2 agents. The relative distance $\Delta_{1,2}$ hence reflects the degree to which the marginal type-1 (type-2) agents are more (less) pessimistic, and in equilibrium it varies to adjust the marginal agents of both types' beliefs about the project value z, so that they both price their respective securities at 1, their opportunity cost of participation.

When security $s_1[\cdot]$ becomes *significantly* more attractive than $s_2[\cdot]$, the limit cutoffs of the two types of agents become distinct from each other, i.e., $\hat{x}_1 < \hat{x}_2$. In this case, $\Delta_{1,2} = \lim_{\sigma \to 0}(\hat{x}_2^\sigma - \hat{x}_1^\sigma)/\sigma = \infty$ so that the potential impact of security design through the strategic channel is exhausted, in the sense that the marginal type-2 (type-1) agents are almost sure that type-1 (type-2) agents will (will not) participate. Although they hold quite opposite beliefs regarding the other type's participation, since security $s_1[\cdot]$ is *significantly* more attractive than $s_2[\cdot]$, the divergence in beliefs cannot close the gap between the attractiveness of the two securities. As a result, the fundamental channel has to come into effect. In particular, since $\hat{x}_1 < \hat{x}_2$, the marginal agents of type-1's (type-2's) belief about the fundamental state θ is strictly more pessimistic (optimistic) than that of the marginal agents of type-2 (type-1). Hence, the marginal agents of type-1 (type-2) are strictly more pessimistic (optimistic) in both the fundamental state and the other type's participation. As such, the equilibrium cutoffs always make the marginal agents of both types price their respective securities at 1.

A.2 Zero Contracting Premium due to Miscoordination

In the baseline model, we assume that the lowest possible project value is always zero irrespective of the state of the economy θ and the level of participation K. As a result, the contracting premium due to miscoordination, which is captured by the difference between the expected security payments to agents and their participation costs, is always positive, and thus the principal always strictly prefers a finer differentiation. In practice, it is also possible that as θ or K gets higher, the lowest possible project value increases accordingly. This opens up the possibility that the principal can use a security bundle to achieve zero contracting premium. Since such a bundle has achieved the theoretically lowest cost, the principal does not benefit from a finer differentiation. In this subsection, we derive the condition for such a security bundle to exist and characterize the minimum number of types required by it.

For any θ and K, define

$$V(\theta, K) \equiv \inf\{z | G(z; \theta, K) > 0\}.$$

The term $V(\theta, K)$ represents the effectively lowest possible project value. For any participation scheme $K\left(\theta; \{(K_t, \theta_t)\}_{t=1}^T\right)$, define

$$\tilde{V}_t(x) \equiv V(\theta_t, x + K_{t-1}) - K_{t-1}$$

and

$$\tilde{V}_t^{(n)}(x) \equiv \underbrace{\tilde{V}_t \circ \tilde{V}_t \circ \cdots \tilde{V}_t}_{n}(x).$$

Proposition 9. *The participation scheme $K\left(\theta; \{(K_t, \theta_t)\}_{t=1}^T\right)$ can be implemented by a finite-type security bundle with zero premium if and only if $\tilde{V}_t(x) > x$ for any $t \in \{1,2,\ldots,T\}$ and $x \in [0, K_t - K_{t-1})$. The minimum number of types to achieve zero premium is $n^* = \sum_{t=1}^T n_t^*$ where $n_t^* \equiv \min\left\{n | \tilde{V}_t^{(n)}(0) \geq K_t - K_{t-1}\right\}$.*

To illustrate the intuition of Proposition 9, we take the types with the lowest cutoff θ_1 as an example and see how n_1^* is determined. Without loss of generality, we assume that this security bundle contains a tranching structure as in Proposition 6. Suppose the first l types have the cutoff θ_1 and they all have zero premium. Zero premium requires that $F_k = \sum_{i=1}^k Q_i$. That means, any marginal type-k agents can receive $F_k - F_{k-1} = Q_k$ in almost all scenarios with positive probability. From the perspective of the marginal type-k agents, at most the first $(k-1)$ types of agents participate almost surely, so the worst scenario with positive probability is no better than only mass $\sum_{j=1}^{k-1} Q_j$ of agents participating. Therefore,

$$V\left(\theta_1, \sum_{j=1}^{k-1} Q_j\right) - F_{k-1} \geq T_{F_k}\left[V\left(\theta_1, \sum_{j=1}^{k-1} Q_j\right)\right] - F_{k-1} \geq Q_k$$
$$\Rightarrow \tilde{V}_1\left(\sum_{j=1}^{k-1} Q_j\right) > \sum_{j=1}^{k} Q_j.$$

Iterating this inequality, we obtain that $\sum_{j=1}^k Q_j \leq \tilde{V}_1^{(k)}(0)$. Therefore, $\tilde{V}_1^{(k)}(0)$ is the upper bound of the mass of the first k types of agents if zero premium is required; n_1^* is the minimum number such that this upper bound exceeds the total mass of the agents with the cutoff θ_1.

Figure 2 illustrates one security bundle that achieves zero premium with $n_1^* = 3$ types. The first type features mass $\tilde{V}_1(0)$ of agents sharing the senior tranche $T_{V(\theta_1,0)}$. Upon observing θ_1, each of them can receive

$$\frac{V(\theta_1, 0)}{\tilde{V}_1(0)} = 1$$

almost surely, so they choose to participate irrespective of others' decisions. The second type features mass $\tilde{V}_1^{(2)}(0) - \tilde{V}_1(0)$ of agents sharing the junior tranche $T_{V(\theta_1,\tilde{V}_1(0))} - T_{V(\theta_1,0)}$. Upon observing θ_1, they know type-1 agents participate almost surely, so each of them can receive

$$\frac{V(\theta_1, \tilde{V}_1(0)) - V(\theta_1, 0)}{\tilde{V}_1^{(2)}(0) - \tilde{V}_1(0)} = 1$$

almost surely. Based on the speculation about type-1 agents, they choose to participate. The n_1^*th

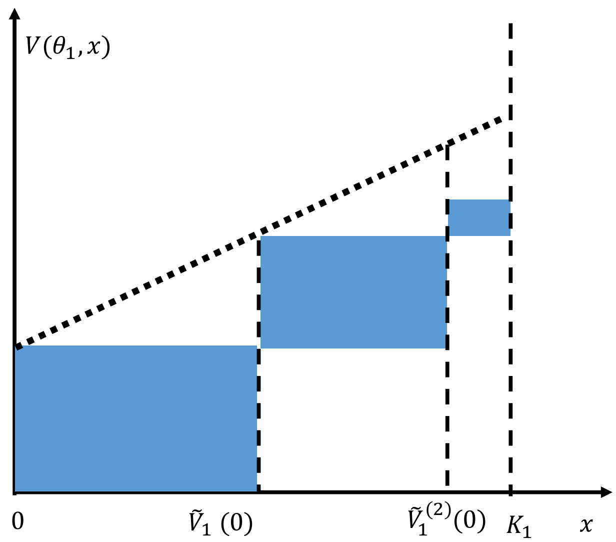

Figure 2: The security bundle achieving zero Premium

type features mass $K_1 - \tilde{V}_1^{(2)}(0)$ of agents sharing the junior tranche $T_{K_1} - T_{V(\theta_1, \tilde{V}_1(0))}$. Conditional on the participation of the first two types of agents, they choose to participate as well.

B Additional Proofs

Proof of Lemma 1

Since $\Phi\left(\Phi^{-1}(1-y)+\Delta_i\right)$ is always between 0 and 1 and strictly increasing in Δ_i, $f(\Delta_i)$ is between 0 and 1 and strictly decreasing in Δ_i.

$$\begin{aligned}
f(\Delta_1)+f(\Delta_2) &= 1-\int_0^1 \Phi\left(\Phi^{-1}(1-y)+\Delta_1\right)dy + 1 - \int_0^1 \Phi\left(\Phi^{-1}(1-y)+\Delta_2\right)dy \\
&= 2-\int_{\infty}^{-\infty} \Phi(x+\Delta_1)\,d(1-\Phi(x)) - \int_{\infty}^{-\infty} \Phi(x)\,d(1-\Phi(x-\Delta_2)) \\
&= 2-\int_{-\infty}^{\infty} \Phi(x+\Delta_1)\,d\Phi(x) - \int_{-\infty}^{\infty} \Phi(x)\,d\Phi(x+\Delta_1) \\
&= 2-\int_{-\infty}^{\infty} d\left[\Phi(x)\Phi(x+\Delta_1)\right] \\
&= 1
\end{aligned}$$

Proof of Lemma 2

Consider $\theta_1 > \theta_2$. $g(z;\theta_1,K)/g(z;\theta_2,K)$ is strictly increasing in z. Let $\tilde{z} \equiv \inf\{g(z;\theta_1,K)/g(z;\theta_2,K) > 1\}$.

$$\begin{aligned}&\left|E\left[S_k[z]|\theta_1,K\right]-E\left[S_k[z]|\theta_2,K\right]\right|\\=&\left|\int_{z>\tilde{z}} S_k[z]\left[g(z;\theta_1,K)-g(z;\theta_2,K)\right]dz-\int_{z<\tilde{z}} S_k[z]\left[g(z;\theta_2,K)-g(z;\theta_1,K)\right]dz\right|\end{aligned}$$

Note that

$$\int_{z>\tilde{z}}\left[g(z;\theta_1,K)-g(z;\theta_2,K)\right]dz=\int_{z<\tilde{z}}\left[g(z;\theta_2,K)-g(z;\theta_1,K)\right]dz.$$

So,

$$\begin{aligned}&\left|E\left[S_k[z]|\theta_1,K\right]-E\left[S_k[z]|\theta_2,K\right]\right|\\=&\frac{\int_{z_1>\tilde{z}}\int_{z_2<\tilde{z}}\left(S_k[z_1]-S_k[z_2]\right)\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]\left[g(z_1;\theta_1,K)-g(z_1;\theta_2,K)\right]dz_2dz_1}{\int_{z_2<\tilde{z}}\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]dz_2}\\\leq&\frac{\int_{z_1>\tilde{z}}\int_{z_2<\tilde{z}}R_k(z_1-z_2)\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]\left[g(z_1;\theta_1,K)-g(z_1;\theta_2,K)\right]dz_2dz_1}{\int_{z_2<\tilde{z}}\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]dz_2}\\=&\left|E\left[R_kz|\theta_1,K\right]-E\left[R_kz|\theta_2,K\right]\right|\\\leq&R_k\left|E\left[z|\theta_1,K\right]-E\left[z|\theta_2,K\right]\right|.\end{aligned}$$

Likewise, for $K_1 > K_2$,

$$\left|E\left[S_k[z]|\theta,K_1\right]-E\left[S_k[z]|\theta,K_2\right]\right|\leq R_k\left|E\left[z|\theta,K_1\right]-E\left[z|\theta,K_2\right]\right|.$$

Then we obtain

$$\begin{aligned}&\left|E\left[S_k[z]|\theta_1,K_1\right]-E\left[S_k[z]|\theta_2,K_2\right]\right|\\=&\left|E\left[S_k[z]|\theta_1,K_1\right]-E\left[S_k[z]|\theta_1,K_2\right]+E\left[S_k[z]|\theta_1,K_2\right]-E\left[S_k[z]|\theta_2,K_2\right]\right|\\\leq&\left|E\left[S_k[z]|\theta_1,K_1\right]-E\left[S_k[z]|\theta_1,K_2\right]\right|+\left|E\left[S_k[z]|\theta_1,K_2\right]-E\left[S_k[z]|\theta_2,K_2\right]\right|\\\leq&R_k\left|E\left[z|\theta_1,K_1\right]-E\left[z|\theta_1,K_2\right]\right|+R_k\left|E\left[z|\theta_1,K_2\right]-E\left[z|\theta_2,K_2\right]\right|.\end{aligned}$$

Proof of Lemma 3

Due to SMLRP,

$$\frac{\phi(\varepsilon+a)}{\phi(\varepsilon+2a)} > \frac{\phi(\varepsilon)}{\phi(\varepsilon+a)}$$
$$\Leftrightarrow -\log\phi(\varepsilon+a) < \frac{-\log\phi(\varepsilon)-\log\phi(\varepsilon+2a)}{2}$$

which means $-\log\phi(\cdot)$ is strictly mid-point convex. Since $\phi(\cdot)$ is continuous, so is $-\log\phi(\cdot)$. According to Jensen (1906), $-\log\phi(\cdot)$ is strictly convex.

Since $\int_{-\infty}^{+\infty}\phi(\varepsilon)d\varepsilon = 1$, $\phi(\cdot)$ cannot be always increasing or decreasing, so $\phi(\cdot)$ must be first increasing and then decreasing. Therefore, $\phi(\cdot)$ is bounded, and $\lim_{|\varepsilon|\to+\infty}\phi(\varepsilon)\to 0$.

Proof of Lemma 4

For type-k agents, the marginal payoff of participating is $E\left[s_k[z]|\theta,K\right]-1$. On the one hand,

$$E\left[s_k[z]|\theta,K\right]-1 = E\left[S_k[z]-S_k[0]|\theta,K\right]/Q_k - 1 \le E\left[z|\theta,1\right]R_k/Q_k - 1.$$

The right-hand side converges to -1 as θ goes to $-\infty$, so there exists $\underline{\theta}$ such that $E\left[s_k[z]|\theta,K\right]-1 < 0$ for $\theta \le \underline{\theta}$ irrespective of K. On the other hand, for $\hat{z}$ satisfying $s_k[\hat{z}] > 1$,

$$E\left[s_k[z]|\theta,K\right]-1 \ge s_k[\hat{z}]\left[1-G(\hat{z};\theta,0)\right]-1.$$

The right-hand side converges to $s_k[\hat{z}]-1$ as θ goes to $+\infty$, so there exists $\overline{\theta}$ such that $E\left[s_k[z]|\theta,K\right]-1 > 0$ for $\theta \ge \overline{\theta}$ irrespective of K.

Proof of Proposition 3

According to the indifference condition of the marginal type-k agents, equation (10), we obtain

$$\int_{-\infty}^{\infty}\left[\int_0^{\infty}(S_k[z]-Q_k)\,g\left(z;\theta,M^{\sigma}(\theta)\right)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)h(\theta)d\theta = 0.$$

Let $m_k = 1-\Phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)$ and $\Delta_{k,j}^{\sigma} \equiv \left(\hat{x}_j^{\sigma}-\hat{x}_k^{\sigma}\right)/\sigma$. Then

$$M^{\sigma}(\theta) = \sum_{j=1}^{n}Q_j - \sum_{j=1}^{n}Q_j\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^{\sigma}\right).$$

The above indifference condition can be rewritten as

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\theta,\{\Delta^{\sigma}_{k,j}\}_{j=1}^n,m_k)dz\right]dm_k = 0,$$

where $\theta = \hat{x}^{\sigma}_k - \sigma\Phi^{-1}(1-m_k)$ and

$$\Gamma(z;\theta,\{\Delta^{\sigma}_{k,j}\}_{j=1}^n,m_k) \equiv g\left(z;\theta,\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j\Phi\left(\Phi^{-1}(1-m_k)+\Delta^{\sigma}_{k,j}\right)\right)h(\theta).$$

Part I: There always exists an infinite subsequence $\{\sigma_m\}_{m=1}^{+\infty}$ converging to 0 such that any $\{\hat{x}^{\sigma_m}_k\}_{m=1}^{+\infty}$ converges to $\hat{x}^0_k \in (-\infty,+\infty)$ and any $\left\{\Delta^{\sigma_m}_{j,k}\right\}_{m=1}^{+\infty}$ converges to $\Delta^0_{j,k}\in[-\infty,+\infty]$. Moreover, $\{(\hat{x}^0_k,\Delta^0_{j,k})\}_{j,k\in\{1,2,\dots,n\}}$ satisfy the equation system in Proposition 3.

The existence of such converging sequences is obvious. By Lemma 4, for sufficiently small σ_m, $\hat{x}^{\sigma_m}_k \in (\underline{\theta},\overline{\theta})$, so $\hat{x}^0_k$ must be finite. In addition, if $\hat{x}^0_k > (<) \hat{x}^0_{k-1}$,

$$\Delta^0_{k-1,k} = \lim_{m\to\infty}\Delta^{\sigma_m}_{k-1,k} = \lim_{m\to\infty}\frac{\hat{x}^{\sigma_m}_k - \hat{x}^{\sigma_m}_{k-1}}{\sigma_m} = \lim_{m\to\infty}\frac{\hat{x}^0_k - \hat{x}^0_{k-1}}{\sigma_m} = +\infty(-\infty),$$

and

$$\Delta^0_{j,k} = \lim_{m\to\infty}\Delta^{\sigma_m}_{j,k} = \lim_{m\to\infty}\sum_{i=j+1}^{k}\Delta^{\sigma_m}_{i-1,i} = \sum_{i=j+1}^{k}\Delta^0_{i-1,i}.$$

To confirm the existence of the solution to the equation system in Proposition 3, we will prove

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}_{j=1}^n,m_k)dz\right]dm_k = 0.$$

Step 1 We claim that for $\varepsilon>0$, there exists $t_1>0$ such that

$$\left|\int_{\left|\Phi^{-1}(1-m_k)\right|\geq t_1}\left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\theta,\{\Delta^{\sigma}_{k,j}\}_{j=1}^n,m_k)dz\right]dm_k\right| < \varepsilon$$

and

$$\left|\int_{\left|\Phi^{-1}(1-m_k)\right|\geq t_1}\left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}^0_k,\{\Delta^{\sigma}_{k,j}\}_{j=1}^n,m_k)dz\right]dm_k\right| < \varepsilon.$$

Note that

$$
\begin{aligned}
&\left|\int_{\left|\Phi^{-1}(1-m_k)\right|\geq t_1}\left[\int_0^{\infty}(S_k[z]-Q_k)\,\Gamma(z;\theta,\{\Delta_{k,j}^{\sigma}\}_{j=1}^n,m_k)dz\right]dm_k\right| \\
&\leq\int_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}\left[\int_0^{\infty}(S_k[z]+Q_k)\,g\left(z;\theta,1\right)h(\theta)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)d\theta \\
&\leq\int_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}\left[E\left[z|\theta,1\right]h(\theta)+Q_kh(\theta)\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)d\theta \\
&\leq\int_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}\left[E\left[z|\theta,1\right]h(\theta)+Q_kh(\theta)\right]d\theta\cdot\frac{1}{\sigma}\sup_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}\left\{\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)\right\} \\
&\leq\left[E\left[z|1\right]+Q_k\right]\cdot\frac{1}{\sigma}\sup_{|y|\geq t_1}\left\{\phi\left(y\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{\left|\Phi^{-1}(1-m_k)\right|\geq t_1}\left[\int_0^{\infty}(S_k[z]-Q_k)\,\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma}\}_{j=1}^n,m_k)dz\right]dm_k\right| \\
&\leq\int_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}\left[\int_0^{\infty}(S_k[z]+Q_k)\,g\left(z;\hat{x}_k^0,1\right)h(\hat{x}_k^0)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)d\theta \\
&\leq\left(E\left[z|\hat{x}_k^0,1\right]+Q_k\right)h(\hat{x}_k^0)\int_{\left|\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right|\geq t_1}d\left[1-\Phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)\right] \\
&\leq\left(E\left[z|\hat{x}_k^0,1\right]+Q_k\right)h(\hat{x}_k^0)\left[1-\Phi\left(t_1\right)+\Phi\left(-t_1\right)\right]
\end{aligned}
$$

Since $\lim_{|y|\to+\infty}\phi(y)=0$ and $\int_{-\infty}^{\infty}\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^{\sigma}-\theta}{\sigma}\right)d\theta$ is finite, as $t_1\to+\infty$,

$$
\begin{gathered}
\frac{1}{\sigma}\sup_{|y|\geq t_1}\left\{\phi\left(y\right)\right\}\to 0, \\
1-\Phi\left(t_1\right)+\Phi\left(-t_1\right)\to 0.
\end{gathered}
$$

Hence, such t_1 exists and does not depend on σ.

Step 2 We claim that there exists $\overline{\sigma}$ such that for any $\sigma_m<\overline{\sigma}$,

$$
\left|\int_{\left|\Phi^{-1}(1-m_k)\right|<t_1}\int_0^{\infty}(S_k[z]-Q_k)\left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)-\Gamma(z;\theta,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)\right]dzdm_k\right|<\varepsilon.
$$

Since $\left|\Phi^{-1}(1-m_k)\right| < t_1$, $\left|\theta - \hat{x}_k^0\right| < \left|\hat{x}_k^0 - \hat{x}_k^{\sigma_m}\right| + \sigma t_1 \equiv t_2$. As $\sigma \to 0$, $t_2 \to 0$. Denote $\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}^{\sigma_m}\right)$ by M.

$$
\begin{aligned}
&\left|\int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)-\Gamma(z;\theta,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)\right]dz\right| \\
\le&\left|\int_0^\infty S_k[z]\left[g\left(z;\hat{x}_k^0,M\right)-g\left(z;\theta,M\right)\right]dz\right|h(\theta)+\int_0^\infty S_k[z]g\left(z;\hat{x}_k^0,M\right)dz\cdot\left|h(\hat{x}_k^0)-h(\theta)\right|+Q_k\left|h(\hat{x}_k^0)-h(\theta)\right| \\
\le&\left|\int_0^\infty S_k[z]\left[g\left(z;\hat{x}_k^0,M\right)-g\left(z;\theta,M\right)\right]dz\right|h(\theta)+E\left[z|\hat{x}_k^0,1\right]\cdot\left|h(\hat{x}_k^0)-h(\theta)\right|+Q_k\left|h(\hat{x}_k^0)-h(\theta)\right|
\end{aligned}
$$

Note that $h(\theta)$ is continuous in θ. As $t_2 \to 0$, $\left|h(\hat{x}_k^0)-h(\theta)\right| \to 0$.

For $\lambda_i(\hat{x}_k^0 - t_2) \le M < \lambda_{i+1}(\hat{x}_k^0 + t_2)$, since $\lambda_i(\hat{x}_k^0) \le M < \lambda_{i+1}(\hat{x}_k^0)$ and $\lambda_i(\theta) \le M < \lambda_{i+1}(\theta)$,

$$
\begin{aligned}
\left|\int_0^\infty S_k[z]\left[g\left(z;\hat{x}_k^0,M\right)-g\left(z;\theta,M\right)\right]dz\right|h(\theta) &= \left|\int_0^\infty S_k[z]\left[g_i\left(z;\hat{x}_k^0,M\right)-g_i\left(z;\theta,M\right)\right]dz\right|h(\theta) \\
&\le R_k\left|\int_0^\infty z\left[g_i\left(z;\hat{x}_k^0,M\right)-g_i\left(z;\theta,M\right)\right]dz\right|h(\theta) \\
&\le R_k R t_2 \cdot \sup\{h(\cdot)\},
\end{aligned}
$$

which converges to 0 as $t_2 \to 0$. For $\lambda_i(\hat{x}_k^0 + t_2) \le M < \lambda_{i+1}(\hat{x}_k^0 - t_2)$,

$$
\begin{aligned}
&\left|\int_0^\infty S_k[z]\left[g\left(z;\hat{x}_k^0,M\right)-g\left(z;\theta,M\right)\right]dz\right|h(\theta) \\
\le& R_k\left|\int_0^\infty z\left[g\left(z;\hat{x}_k^0,M\right)-g\left(z;\theta,M\right)\right]dz\right|\cdot\sup\{h(\cdot)\} \\
\le& R_k\left|E\left[z|\hat{x}_k^0,M\right]-E\left[z|\theta,M\right]\right|\cdot\sup\{h(\cdot)\},
\end{aligned}
$$

which converges to 0 as $t_2 \to 0$, since $\left|E\left[z|\hat{x}_k^0,M\right]-E\left[z|\theta,M\right]\right|$ is bounded and the measure of m_k satisfying $\lambda(\hat{x}_k^0 + t_2) \le M < \lambda(\hat{x}_k^0 - t_2)$ goes to 0 as $t_2 \to 0$.

Taken together, such $\overline{\sigma}$ exists.

Step 3 We claim that there exists $\overline{\sigma}$ such that for any $\sigma_m < \overline{\sigma}$,

$$
\left|\int_{|\Phi^{-1}(1-m_k)|<t_1}\int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)-\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k)\right]dzdm_k\right| < \varepsilon.
$$

Here we prove that as $\sigma_m \to 0$,

$$
\left|\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^{\sigma_m}\right)-\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right)\right|
$$

uniformly converge to 0 for any m_k satisfying $\left|\Phi^{-1}(1-m_k)\right| < t_1$. If $\Delta^0_{k,j}$ is finite, it follows that $\Delta^{\sigma_m}_{k,j} \to \Delta^0_{k,j}$ and $\phi(\cdot)$ is bounded. If $\Delta^0_{k,j} = +\infty$, $\Phi\left(\Phi^{-1}(1-m_k)+\Delta^0_{k,j}\right) = 1$. Since $\Phi\left(\Phi^{-1}(1-m_k)+\Delta^{\sigma}_{k,j}\right)$ increases to 1 as $\Delta^{\sigma_m}_{k,j} \to +\infty$, for any $\delta > 0$, when σ_m is sufficiently small, $\Delta^{\sigma_m}_{j,k}$ can be large enough such that for any m_k satisfying $\left|\Phi^{-1}(1-m_k)\right| < t_1$,

$$\Phi\left(\Phi^{-1}(1-m_k)+\Delta^{\sigma_m}_{k,j}\right) \geq \Phi(-t_1+\Delta^{\sigma_m}_{k,j}) > 1-\delta.$$

The case of $\Delta^0_{k,j} = -\infty$ follows a similar proof.

Step 4 For $\sigma_m < \overline{\sigma}$,

$$\begin{aligned}
&\left|\int_0^1 \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}^0_k,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)-\Gamma(z;\theta,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)\right]dzdm_k\right| \\
\leq &\left|\int_{|\Phi^{-1}(1-m_k)|<t_1} \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}^0_k,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)-\Gamma(z;\theta,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)\right]dzdm_k\right| + 2\varepsilon \\
\leq &3\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left|\int_0^1 \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}^0_k,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)-\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}^n_{j=1},m_k)\right]dzdm_k\right| \\
\leq &\left|\int_{|\Phi^{-1}(1-m_k)|<t_1} \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}^0_k,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)-\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}^n_{j=1},m_k)\right]dzdm_k\right| + 2\varepsilon \\
< &3\varepsilon.
\end{aligned}$$

Taken together,

$$\begin{aligned}
&\left|\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}^n_{j=1},m_k)dz\right]dm_k\right| \\
= &\left|\int_0^1 \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}^n_{j=1},m_k)-\Gamma(z;\theta,\{\Delta^{\sigma_m}_{k,j}\}^n_{j=1},m_k)\right]dzdm_k\right| \\
< &6\varepsilon.
\end{aligned}$$

Since ε can be arbitrarily small,

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}^0_k,\{\Delta^0_{k,j}\}^n_{j=1},m_k)dz\right]dm_k = 0.$$

Let

$$M = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j\Phi\left(\Phi^{-1}(1-m_k)+\Delta^0_{k,j}\right),$$

which ranges from 0 to $\sum_{j=1}^{n} Q_j$.

$$\frac{dm_k}{dM} = \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^{n} Q_j \phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right)} = f\left(M; \{Q_j, \Delta_{k,j}^0\}_{j=1}^n\right).$$

So,

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\, g\left(z; \hat{x}_k^0, M\right) dz\right] f\left(M; \{Q_j, \Delta_{k,j}^0\}_{j=1}^n\right) dM = 0.$$

We have confirmed the existence of the solution to the equation system.

Part II: There is a unique solution $\{\hat{x}_k\}_{k=1}^n$ to the equation system.

Suppose $\{\hat{x}_k\}_{k=1}^n$ and $\{\hat{x}_k'\}_{k=1}^n$ both satisfy the equation system and they are different in at least one element. They have $\{\Delta_{k,j}\}_{j,k\in\{1,\ldots,n\}}$ and $\{\Delta_{k,j}'\}_{j,k\in\{1,\ldots,n\}}$ respectively.

Suppose there are types with $\hat{x}_k' > \hat{x}_k$ and they constitute the set $\mathscr{T} = \{\tau_1, \tau_2, \ldots, \tau_L\}$ where $\tau_1 < \tau_2 \ldots < \tau_L$. Consider $k \in \mathscr{T}$. Since

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\, \Gamma(z; \hat{x}_k', \{\Delta_{k,j}'\}_{j=1}^n, m_k) dz\right] dm_k$$

is strictly decreasing in $\Delta_{k,j}'$, if $\Delta_{k,j}' \leq \Delta_{k,j}$ for any j, then

$$\begin{aligned}
&\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\, \Gamma(z; \hat{x}_k', \{\Delta_{k,j}'\}_{j=1}^n, m_k) dz\right] dm_k \\
&> \int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\, \Gamma(z; \hat{x}_k, \{\Delta_{k,j}\}_{j=1}^n, m_k) dz\right] dm_k \cdot \frac{h(\hat{x}_k')}{h(\hat{x}_k)} = 0.
\end{aligned}$$

Therefore, $\Delta_{k,j}' > \Delta_{k,j}$ for some j. Let $a(k)$ be the first j such that $\Delta_{k,j}' > \Delta_{k,j}$.

First, consider $k = \tau_1$. Note that for $j \notin \mathscr{T}$, since $\hat{x}_j' \leq \hat{x}_j$ and $\hat{x}_k' > \hat{x}_k$, $\Delta_{k,j}' < \Delta_{k,j}$. So, $a(\tau_1) \in \mathscr{T}$ and $a(\tau_1) > \tau_1$. Second, consider $k = a(\tau_1)$. Likewise, $a^{(2)}(\tau_1) = a(a(\tau_1))$ must be in $\mathscr{T}$. By the definition of $a(\tau_1)$, for any $j \in \mathscr{T}$ and $j < a(\tau_1)$, $\Delta_{\tau_1,j}' \leq \Delta_{\tau_1,j}$, and $\Delta_{\tau_1,a(\tau_1)}' > \Delta_{\tau_1,a(\tau_1)}$. So, for these j,

$$\Delta_{a(\tau_1),j}' = \Delta_{\tau_1,j}' - \Delta_{\tau_1,a(\tau_1)}' < \Delta_{\tau_1,j} - \Delta_{\tau_1,a(\tau_1)} = \Delta_{a(\tau_1),j},$$

which implies $a(a(\tau_1)) > a(\tau_1)$. Iterating the procedure, we end up with an infinite sequence $\{a^{(m)}(\tau_1)\}_{m=1}^{+\infty}$ in $\mathscr{T}$. This is impossible because $\mathscr{T}$ is a finite set.

Therefore, the types with $\hat{x}_k' > \hat{x}_k$ do not exist; nor do the types with $\hat{x}_k' < \hat{x}_k$. The solution is unique. Note that the solution is the limits $\{\hat{x}_k^0\}_{k=1}^n$ in Part I.

Part III: The equation system is the necessary and sufficient condition for $\{\hat{x}_k\}_{k=1}^n$ to be the limits of the cutoffs as $\sigma \to 0$.

Suppose as $\sigma \to 0$, $\{\hat{x}_k^\sigma\}_{k=1}^n$ do not converge to $\{\hat{x}_k^0\}_{k=1}^n$. That means, there exists ε and an infinite sequence $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\max_k \left|\hat{x}_k^{\sigma_m} - \hat{x}_k^0\right| > \varepsilon$. According to Part I and Part II, there exists an infinite subsequence $\{\sigma_m\}_{m=1}^{+\infty}$ of $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\{\hat{x}_k^{\sigma_m}\}_{k=1}^n$ converges to $\{\hat{x}_k^0\}_{k=1}^n$, which is impossible. Therefore, as $\sigma \to 0$, $\{\hat{x}_k^\sigma\}_{k=1}^n$ converge to $\{\hat{x}_k^0\}_{k=1}^n$.

Conversely, if $\{\hat{x}_k\}_{k=1}^n$ satisfy the equation system, when the security bundle $\{(S_k, Q_k)\}_{k=1}^n$ is issued, as $\sigma \to 0$, $\{\hat{x}_k^\sigma\}_{k=1}^n$ must converge to the solution of the equation system, which is uniquely $\{\hat{x}_k\}_{k=1}^n$.

Proof of Proposition 4

In the state θ, mass $Q_k m_k^\sigma(\theta)$ of type-k agents accept their offers, so the principal's expected payoff is

$$E[\pi^P] = \int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta))h(\theta)d\theta,$$

where

$$\begin{aligned}
&\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \\
\equiv &\int_0^{+\infty} \left(z - \sum_{k=1}^n m_k^\sigma(\theta) Q_k s_k[z]\right) g\left(z; \theta, \sum_{k=1}^n Q_k m_k^\sigma(\theta\right) dz \\
= &E\left[z|\theta, \sum_{k=1}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=1}^n m_k^\sigma(\theta) E\left[S_k[z]|\theta, \sum_{k=1}^n Q_k m_k^\sigma(\theta)\right].
\end{aligned}$$

For any $\varepsilon > 0$, there exists $t_1 > 0$ such that for any $\theta < \hat{x}_k^\sigma - t_1\sigma$,

$$m_k^\sigma(\theta) = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right) < 1 - \Phi(t_1) < \varepsilon,$$

and for any $\theta > \hat{x}_k^\sigma + t_1\sigma$,

$$m_k^\sigma(\theta) = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right) > 1 - \Phi(-t_1) > 1 - \varepsilon.$$

Consider σ that is sufficiently small such that $\hat{x}_k^\sigma - t_1\sigma > \hat{x}_k - 1$ and $\hat{x}_k^\sigma + t_1\sigma < \hat{x}_k + 1$.

Consider $m_1^\sigma(\theta)$ and $1\{\theta > \hat{x}_1\}$. For $\theta < \underline{\hat{x}_1^\sigma} \equiv \min\{\hat{x}_1^\sigma - t_1\sigma, \hat{x}_1\}$,

$$\begin{aligned}
&\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \\
<&E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^{n} m_k^\sigma(\theta) E\left[S_k[z]|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] dz \\
\leq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) + E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right],
\end{aligned}$$

so

$$\begin{aligned}
&\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta \\
<&\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \left\{ E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] \right\} h(\theta) d\theta.
\end{aligned}$$

For θ satisfying $\lambda_i(\theta) \leq \sum_{k=2}^{n} Q_k m_k^\sigma(\theta) < \lambda_{i+1}(\theta) - Q_1\varepsilon$,

$$\begin{aligned}
&E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] \\
=&E_i\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E_i\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] \\
\leq&RQ_1\varepsilon.
\end{aligned}$$

For θ satisfying $\lambda_{i+1}(\theta) - Q_1\varepsilon \leq \sum_{k=2}^{n} Q_k m_k^\sigma(\theta) < \lambda_{i+1}(\theta)$,

$$E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right]$$

is bounded. Therefore, as $\varepsilon \to 0$,

$$\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \left\{ E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^{n} Q_k m_k^\sigma(\theta)\right] \right\} h(\theta) d\theta$$

can be arbitrarily small.

On the other hand, for $\theta < \underline{\hat{x}_1^\sigma}$,

$$\begin{aligned}
&\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \\
>&E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \varepsilon E\left[S_1[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \\
\geq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) - \varepsilon E[z|\theta, 1] \\
&-\left\{\sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right]\right\} \\
\geq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) - \varepsilon E[z|\theta, 1] - \sum_{k=2}^n R_k \left\{E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right]\right\}.
\end{aligned}$$

Following the same argument, we obtain that as $\varepsilon \to 0$,

$$\left|\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right|$$

can be arbitrarily small.

Likewise, let $\overline{\hat{x}_1^\sigma} \equiv \max\{\hat{x}_1^\sigma + t_1\sigma, \hat{x}_1\}$. As $\varepsilon \to 0$,

$$\left|\int_{\overline{\hat{x}_1^\sigma}}^{+\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{\overline{\hat{x}_1^\sigma}}^{+\infty} \Pi(1, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right|$$

can be arbitrarily small.

For $\theta \in [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}] \subseteq [\hat{x}_1 - 1, \hat{x}_1 + 1]$,

$$0 \leq \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \leq E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right],$$

so

$$\begin{aligned}
&\left|\int_{\underline{\hat{x}_1^\sigma}}^{\overline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{\underline{\hat{x}_1^\sigma}}^{\overline{\hat{x}_1^\sigma}} \Pi(1\{\theta > \hat{x}_1\}, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right| \\
\leq&2E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right] \sup\{h(\cdot)\} \cdot \left(\overline{\hat{x}_1^\sigma} - \underline{\hat{x}_1^\sigma}\right) \\
\leq&2E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right] \sup\{h(\cdot)\} \cdot \left(|\hat{x}_1 - \hat{x}_1^\sigma| + 2t_1\sigma\right),
\end{aligned}$$

which converges to 0 as $\sigma \to 0$.

To sum up, for any $\delta > 0$, there exists $\overline{\sigma}_1$ such that for any $\sigma < \overline{\sigma}_1$,

$$\left|\int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right| < \delta.$$

Repeating the analysis on all $k \in \{1,2,\ldots,n\}$, we obtain as $\sigma \to 0$,

$$\int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta))h(\theta)d\theta \to \int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, 1\{\theta > \hat{x}_2\}, \ldots, 1\{\theta > \hat{x}_n\})h(\theta)d\theta.$$

It is easy to see

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, 1\{\theta > \hat{x}_2\}, \ldots, 1\{\theta > \hat{x}_n\})h(\theta)d\theta \\
&= \sum_{k=0}^{n} \int_{\hat{x}_k}^{\hat{x}_{k+1}} \left[\int_0^{+\infty} \left(z - \sum_{j=1}^{k} Q_j s_j[z] \right) g\left(z; \theta, \sum_{j=1}^{k} Q_j \right) dz \right] h(\theta) d\theta,
\end{aligned}$$

where $\hat{x}_0 = -\infty$ and $\hat{x}_{n+1} = +\infty$.

Next, we show that $\sum_{k=1}^{n} Q_k m_k^\sigma(\theta)$ converges to $\sum_{k=1}^{n} Q_k \cdot 1\{\theta \geq \hat{x}_k\}$ in probability. Consider $m_1^\sigma(\theta)$ and $1\{\theta \geq \hat{x}_1\}$. According to the above construction, for $\theta \notin [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}]$, $\left| m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\} \right| < \varepsilon$, so

$$Pr\left[\left|m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\}\right| > \varepsilon\right] \leq Pr\left[\theta \in [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}]\right] \leq \sup\{h(\cdot)\} \cdot \left(|\hat{x}_1 - \hat{x}_1^\sigma| + 2t_1\sigma\right).$$

That means, $Pr\left[\left|m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\}\right| > \varepsilon\right]$ can be arbitrarily small when σ is sufficiently small. Similarly, this argument applies to all other k. Therefore,

$$\lim_{\sigma \to 0} Pr\left[\left| \sum_{k=1}^{n} Q_k m_k^\sigma(\theta) - \sum_{k=1}^{n} Q_k \cdot 1\{\theta \geq \hat{x}_k\} \right| > \varepsilon \right] = 0.$$

Proof of Lemma 5

Denote the alternative security bundle as $\{(S'_k, Q'_k)\}_{k=1}^{n}$ where

$$S'_k = \begin{cases} S_i + S_{i+1} & \text{if } k = i \\ 0 & \text{if } k = i+1 \\ S_k, & \text{otherwise} \end{cases}$$

and

$$Q'_k = \begin{cases} Q_i + Q_{i+1} & \text{if } k = i \\ 0 & \text{if } k = i+1 \\ Q_k, & \text{otherwise} \end{cases}.$$

$(S'_{i+1}, Q'_{i+1}) = (0,0)$ is used only for notational convenience and can be ignored. We only need to check that the condition in Proposition 3 holds for the alternative security bundle with $\{\hat{x}_k, \Delta_{j,k}\}_{j,k \in \{1,2,\ldots,n\}}$. Since $\Delta_{i,i+1} = 0$, $\Delta_{i,k} = \Delta_{i+1,k}$ and $\hat{x}_i = \hat{x}_{i+1}$.

$$
\begin{aligned}
& f\left(M;\{Q'_j,\Delta_{k,j}\}_{j=1}^n\right) \\
=& \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q'_j \phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right)} \\
=& \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j\neq i, j\neq i+1} Q_j \phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right)+(Q_i+Q_{i+1})\phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,i}\right)} \\
=& \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q_j \phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right)},
\end{aligned}
$$

where

$$
M=\sum_{j=1}^n Q'_j-\sum_{j=1}^n Q'_j \Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right)=\sum_{j=1}^n Q_j-\sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}\right).
$$

So,

$$
f\left(M;\{Q'_j,\Delta_{k,j}\}_{j=1}^n\right)=f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^n\right).
$$

That means, for k other than i or $i+1$,

$$
\begin{aligned}
& \int_0^\infty S'_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k,M\right) f\left(M;\{Q'_j,\Delta_{k,j}\}_{j=1}^n\right) dM\right] dz \\
=& \int_0^\infty S_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k,M\right) f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^n\right) dM\right] dz \\
=& Q_k
\end{aligned}
$$

For $k=i$,

$$
\begin{aligned}
& \int_0^\infty S'_i[z]\left[\int_0^\infty g\left(z;\hat{x}_i,M\right) f\left(M;\{Q'_j,\Delta_{i,j}\}_{j=1}^n\right) dM\right] dz \\
=& \int_0^\infty \left(S_i[z]+S_{i+1}[z]\right)\left[\int_0^\infty g\left(z;\hat{x}_i,M\right) f\left(M;\{Q_j,\Delta_{i,j}\}_{j=1}^n\right) dM\right] dz \\
=& Q_i+Q_{i+1}
\end{aligned}
$$

Therefore, for all types, the condition in Proposition 3 holds.

The proof of the converse is similar.

Proof of Lemma 6

Consider $W^P(z;\hat{x}_k,b)/W_k^A(z)$.

$$\frac{W^P(z;\hat{x}_k,b)}{W_k^A(z)} = \int_{\hat{x}_k}^{b} \frac{g\left(z;\theta,K\left(\theta\right)\right)}{\int_0^\infty g\left(z;\hat{x}_k,M\right) f_k\left(M\right) dM} h(\theta)d\theta = \int_{\hat{x}_k}^{b} \frac{1}{\int_0^\infty \frac{g(z;\hat{x}_k,M)}{g(z;\theta,K(\theta))} f_k\left(M\right) dM} h(\theta)d\theta.$$

Since $\hat{x}_k < \theta$ and $M \leq K\left(\theta\right)$, $g\left(z;\hat{x}_k,M\right)/g\left(z;\theta,K\left(\theta\right)\right)$ is strictly decreasing in z. Also, $f_k\left(M\right)$ is positive for a positive measure of M. Therefore $W^P(z;\hat{x}_k,b)/W_k^A(z)$ is strictly increasing in z.

Consider $W_k^A(z)/W_{k-1}^A(z)$. If $\Delta_{k-1,k} = +\infty$, $U(k-1) = L(k)$.

$$\frac{W_k^A(z)}{W_{k-1}^A(z)} = \frac{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_k,M\right) f_k\left(M\right) dM}{\int_{\sum_{j=1}^{L(k-1)} Q_j}^{\sum_{j=1}^{U(k-1)} Q_j} g\left(z;\hat{x}_{k-1},y\right) f_{k-1}\left(y\right) dy} = \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(M\right)}{\int_{\sum_{j=1}^{L(k-1)} Q_j}^{\sum_{j=1}^{U(k-1)} Q_j} \frac{g(z;\hat{x}_{k-1},y)}{g(z;\hat{x}_k,M)} f_{k-1}\left(y\right) dy} dM.$$

Since $M > y$ and $\hat{x}_k \geq \hat{x}_{k-1}$, $g\left(z;\hat{x}_{k-1},y\right)/g\left(z;\hat{x}_k,M\right)$ is strictly decreasing in z. Therefore, $W_k^A(z)/W_{k-1}^A(z)$ is strictly increasing in z.

If $\Delta_{k-1,k} < +\infty$, let

$$\Omega(z,y) \equiv \int_{\sum_{j=1}^{L(k)} Q_j}^{y} \frac{g\left(z;\hat{x}_{k-1},M\right)\cdot f_{k-1}\left(M\right)}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_{k-1},M\right)\cdot f_{k-1}\left(M\right) dM} dM.$$

$\Omega(z,y)$ is strictly decreasing in z for any $y \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ because

$$\Omega(z,y) = \frac{1}{1 + \frac{\int_y^{\sum_{j=1}^{U(k)} Q_j} g(z;\hat{x}_{k-1},M)\cdot f_{k-1}(M)dM}{\int_{\sum_{j=1}^{L(k)} Q_j}^{y} g(z;\hat{x}_{k-1},M)\cdot f_{k-1}(M)dM}}.$$

We have

$$\begin{aligned}\frac{W_k^A(z)}{W_{k-1}^A(z)} &= \frac{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_k,y\right)\cdot f_k\left(y\right) dy}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_{k-1},M\right)\cdot f_{k-1}\left(M\right) dM} \\ &= \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(y\right)}{f_{k-1}\left(y\right)} \frac{g\left(z;\hat{x}_k,y\right) f_{k-1}\left(y\right)}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{x}_{k-1},M\right)\cdot f_{k-1}\left(M\right) dM} dy = \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(y\right)}{f_{k-1}\left(y\right)} d\Omega(z,y).\end{aligned}$$

By $\Omega\left(z,\sum_{j=1}^{L(k)} Q_j\right) = 0$ and $\Omega\left(z,\sum_{j=1}^{U(k)} Q_j\right) = 1$, using integration by parts, we obtain

$$\frac{W_k^A(z)}{W_{k-1}^A(z)} = \frac{f_k\left(\sum_{j=1}^{U(k)} Q_j\right)}{f_{k-1}\left(\sum_{j=1}^{U(k)} Q_j\right)} - \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \Omega(z,y) d\left[\frac{f_k(y)}{f_{k-1}(y)}\right].$$

By the fifth property in Proposition 5, $f_k(y)/f_{k-1}(y)$ is strictly increasing in y and $\Omega(z,y)$ is strictly decreasing in z over $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$. So, $W_k^A(z)/W_{k-1}^A(z)$ is strictly increasing in z.

Proof of Proposition 9

Part I: $\tilde{V}_t(x) > x$ is a necessary condition for zero premium.

Suppose $K\left(\theta; \{(K_t, \theta_t)\}_{t=1}^T\right)$ can be implemented by a finite-type security bundle with zero premium and there exist $\hat{t} \in \{1,2,\ldots,T\}$ and $\hat{x} \in [0, K_{\hat{t}} - K_{\hat{t}-1})$ such that $\tilde{V}_{\hat{t}}(\hat{x}) \leq \hat{x}$, i.e.,

$$V\left(\theta_{\hat{t}}, \hat{x} + K_{\hat{t}-1}\right) \leq \hat{x} + K_{\hat{t}-1}$$

Suppose the security bundle is $\{(S_k, Q_k)\}_{k=1}^n$. Consider the kth type with the cutoff $\theta_{\hat{t}}$ such that $\sum_{j=1}^{k-1} Q_j \leq \hat{x} + K_{\hat{t}-1} < \sum_{j=1}^{k} Q_j$. On the one hand, because of zero premium,

$$V\left(\theta_{\hat{t}}, \sum_{j=1}^{L(k)} Q_j\right) \geq \sum_{j=1}^{k} Q_j.$$

On the other hand, since $L(k) \leq k-1$ and $\tilde{V}_t(x)$ is increasing in x,

$$V\left(\theta_{\hat{t}}, \sum_{j=1}^{L(k)} Q_j\right) \leq V\left(\theta_{\hat{t}}, \sum_{j=1}^{k-1} Q_j\right) \leq V\left(\theta_{\hat{t}}, \hat{x} + K_{t-1}\right) \leq \hat{x} + K_{\hat{t}-1}.$$

So,

$$\sum_{j=1}^{k} Q_j \leq= \hat{x} + K_{\hat{t}-1}.$$

Contradiction! Therefore, such $\hat{t}$ and $\hat{x}$ cannot exist.

Part II: $\tilde{V}_t(x) > x$ is a sufficient condition for the participation scheme to have an n^*-type security bundle achieving zero premium.

Suppose that $\tilde{V}_t(x) > x$ for any $t \in \{1,2,\ldots,T\}$ and $x \in (0, K_t - K_{t-1})$.

First, we prove the existence of n_t^*. Suppose not. Then $\tilde{V}_t^{(n)}(0) < K_t - K_{t-1}$ for any n. Since $\tilde{V}_t(x)/x$ is continuous over $[\tilde{V}_t(0), K_t - K_{t-1}]$, there exists $\underline{x} \in [\tilde{V}_t(0), K_t - K_{t-1}]$ such that $\tilde{V}_t(x)/x \geq$

$\tilde{V}_t(\underline{x})/\underline{x}$. Notice

$$\frac{\tilde{V}_t^{(n)}(0)}{K_t - K_{t-1}} = \frac{\tilde{V}_t(0)}{K_t - K_{t-1}} \prod_{k=2}^{n} \frac{\tilde{V}_t\left(\tilde{V}_t^{(k-1)}(0)\right)}{\tilde{V}_t^{(k-1)}(0)}$$

and $\tilde{V}_t^{(k-1)}(0) \in [\tilde{V}_t(0), K_t - K_{t-1}]$. So,

$$\frac{\tilde{V}_t^{(n)}(0)}{K_t - K_{t-1}} \geq \frac{\tilde{V}_t(0)}{K_t - K_{t-1}} \left[\frac{\tilde{V}_t(\underline{x})}{\underline{x}}\right]^{n-1}.$$

Since $\tilde{V}_t(\underline{x})/\underline{x} > 1$, when n is sufficiently large, $\tilde{V}_t^{(n)}(0) > K_t - K_{t-1}$. Contradiction!

Second, we construct an n^*-type security bundle achieving zero premium. For the cutoff θ_t, the n_t^* contracts are represented by

$$\left\{\left(S_{k+\sum_{i=1}^{t-1} n_i^*}, Q_{k+\sum_{i=1}^{t-1} n_i^*}\right)\right\}_{k=1}^{n_t^*}$$

where for $k < n_i^*$

$$S_{k+\sum_{i=1}^{t-1} n_i^*} = T_{\tilde{V}_t^{(k)}(0)+K_{t-1}} - T_{\tilde{V}_t^{(k-1)}(0)+K_{t-1}}$$
$$Q_{k+\sum_{i=1}^{t-1} n_i^*} = \tilde{V}_t^{(k)}(0) - \tilde{V}_t^{(k-1)}(0),$$

and

$$S_{\sum_{i=1}^{t} n_i^*} = T_{K_t} - T_{\tilde{V}_t^{(n_t^*-1)}(0)+K_{t-1}}$$
$$Q_{\sum_{i=1}^{t} n_i^*} = K_t - K_{t-1} - \tilde{V}_t^{(n_t^*-1)}(0).$$

It is straightforward to see this n^*-type security bundle can implement the participation scheme with zero premium and all $\Delta_{k-1,k}$ being $+\infty$.

Part III: any security bundle with fewer than n^* types cannot achieve zero premium.

Suppose $\{(S_k, Q_k)\}_{k=1}^{n}$ with $\Delta_{k-1,k} > 0$ can achieve zero premium and there are l types with the cutoff θ_l: $\tau+1, \tau+2, \ldots, \tau+l$. According to Proposition 6, it is without loss of generality to assume $S_k \equiv T_{F_k} - T_{F_{k-1}}$. Because of zero premium,

$$\begin{aligned}\sum_{j=\tau+1}^{\tau+k} Q_j &\leq V\left(\theta_t, \sum_{j=\tau+1}^{\tau+k-1} Q_j + K_{t-1}\right) - K_{t-1} \\ &= \tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+k-1} Q_j\right).\end{aligned}$$

Since $\tilde{V}_t(x)$ is increasing in x,

$$K_t - K_{t-1} = \sum_{j=\tau+1}^{\tau+l} Q_j \leq \tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+l-1} Q_j\right) \leq \tilde{V}_t\left(\tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+l-2} Q_j\right)\right) = \tilde{V}_i^{(l)}(0).$$

By the definition of n_i^*, $l \geq n_i^*$. Therefore, to achieve zero premium, the whole bundle must have at least n^* types.