The Optimal Structure of Securities under Coordination Frictions*

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Abstract

We study multi-agent security design in the presence of coordination frictions. A principal intends to develop a project whose value increases with an unknown state and the level of agents' participation. To motivate the participation of ex ante homogeneous agents, the principal offers them multiple monotone securities backed by the project value. More participation results in a higher project value and thus higher security payment to participating agents, making participation decisions strategic complements. Miscoordination arises because agents cannot precisely infer others' decisions from noisy signals about the state. We identify two objects in security design—"payoff sensitivity" and "perception of participation"—that determine the impact of miscoordination. To mitigate the adverse impact of miscoordination, the two objects should be matched assortatively over agents. This mechanism implies a multi-tranche security structure in which senior-tranche holders are more robust to potential miscoordination and participate more aggressively, helping alleviate the junior-tranche holders' fear of miscoordination. We find that the principal's ability to differentiate agents in security format is crucial to whether differentiation is desirable.

Keywords: contracting with externalities, coordination frictions, security design, global games

JEL Codes: D71, D83, D86, G32, L24

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1 Introduction

Project implementation often requires the participation of multiple agents. For example, an entrepreneur needs to raise capital from multiple investors, a financial system relies on multiple interconnected financial institutions and investors to operate, and a firm organizes multiple divisions to work on a joint task. In such settings, each agent's payoff is typically contingent on the project outcome, which depends on the participation decisions of all agents.[1] As a result, agents care about others' participation decisions, and coordination is crucial. However, coordination can be impeded for various reasons. For instance, coordination through communication may be too costly, especially between a large number of agents; it may take a long time to coordinate while participation decisions are urgent; communication itself may be vague and ineffective due to different interpretations. As a result, the agents face strategic uncertainty between each other and may miscoordinate in their decisions, making them reluctant to participate. Such reluctance is self-fulfilling and self-reinforcing, making it even harder for the principal to motivate the agents to participate and achieve efficient outcomes.

This paper studies the principal's optimal security design in the presence of coordination frictions. The principal has the flexibility to design securities and offer different securities to different agents, which enables the study of the allocation of payoffs across states and agents, as well as their interaction. The analysis centers on three key questions: which security format is the most effective in addressing miscoordination, whether agents should be distinguished based on security format or pricing, and what mechanism determines the desirability of a particular way of differentiating the agents.

To address these questions in a unified framework, we formulate the problem in the context of joint task, where a risk-neutral principal ("she") aims to develop a project whose value increases in an unknown state and the participation of multiple risk-neutral agents ("he").[2] The principal offers monotone securities, backed by the project's value, to motivate agents' participation. In this setting, the more agents participate, the higher the project value is, leading to higher security payments for all agents, thereby making their participation decisions strategic complements. The agents receive private noisy signals about the state before deciding whether to participate, resulting in potential miscoordination due to imprecise inference of others' decisions from their signals. As such, we adopt the global games approach and study the principal's multi-agent security design problem on top of it.

[1] Here, we use the term "participation" in a general sense to refer to an agent's action that contributes to the overall outcome of the project. Depending on the context, this could refer to investing capital, exerting effort, or contributing ideas or resources, etc.

[2] We will use she/her to refer to the principal and he/his to (one of) the agents throughout the paper. We do not intentionally associate the players with particular genders.

To analyze the impact of security design on agents' participation, note that in the sub-game of participation each agent adopts a cutoff strategy, participating only if his signal exceeds his cutoff. Therefore, understanding the security design's impact on the marginal agents, those whose signals are equal to their cutoffs, is crucial. Two objects in security design are found to be important. The first is a marginal agent i's *perception of participation*, $f^i(M)$, which represents the probability that mass M of agents participate from his perspective. Similar to the belief constraint in Sákovics and Steiner (2012), the total perception of participation of all agents is fixed, i.e.,

$$\int_i f^i(M)di = 1,$$

which is an important constraint on security design.[3] In addition, a marginal agent with a lower cutoff has a lower perception of participation since he expects other agents to observe lower signals and, consequently, be less likely to participate. Therefore, security design essentially alters the distribution of perception of participation among agents. The second object is an agent's *payoff sensitivity* to the project value, which captures how his expected payoff increases with other agents' participation. Since the agents' total payoffs cannot exceed the project value, this imposes a constraint on the total payoff sensitivity of all agents.

As a result of the complementarity in production technology, agents' expected payoffs increase with the expected participation of other agents, allowing the principal to economize on security offering. We find that the benefits from the complementarity amount to the product of the perception of participation and the payoff sensitivity, summed over all agents. To maximize total benefits while adhering to the aforementioned constraints on the total perception of participation and total payoff sensitivity, the principal should differentiate agents in both objects and achieve an assortative matching between them when designing the security. Based on this mechanism, we derive the qualitative properties of the optimal security design. We assume that the principal can offer a security bundle with at most N formats of securities and focus on the optimal bundle with the fewest formats.

To understand the mechanism, we start by fixing agents' perception of participation and determine the optimal security formats. We find that the optimal security bundle must be structured in tranches, with agents who have a lower perception of participation receiving more senior tranches, and agents who have the same perception of participation receiving identical tranches. This is because the principal and the marginal agents value cash flows differently. Firstly, relative to the marginal agents, the principal cares more (less) about the security payments when the project value is high (low). Since any agent with a signal higher than his cutoff also participates, when the prin-

[3] When the agents have the same security payment, the sub-game of participation becomes a standard global game where the players have symmetric payoffs and play the same cutoff strategy in equilibrium, so that the above constraint reduces to the usual Laplacian belief (i.e., all marginal agents i perceives $f^i(M) = 1$, the uniform distribution over $[0,1]$.

cipal makes security payments to an agent, she conditions on that the project value is overall higher than what a marginal agent expects. Thus, the principal is willing to give the agents (as a whole) a senior tranche and retain the junior tranche for herself, creating a tranching structure between the principal and the agents. Secondly, marginal agents with a lower perception of participation attach more weights to security payments at low project values since they observe lower signals and view low project values as more likely. To optimize security offerings, the principal should allocate cash flows at specific project values to agents who value them the most, leading to a tranching structure within agents based on their perception of participation. Since the agents with the same perception of participation value cash flows in the same way, the principal does not benefit from differentiating them and can offer them an identical tranche.

Given that the tranching structure is based on agents' perception of participation, we next study the extent to which agents should be differentiated in this dimension. We find that a finer differentiation is always preferred, and the optimal security bundle should contain the maximum number of tranches. Firstly, senior-tranche holders benefit less from others' participation since they have lower payoff sensitivity overall than junior-tranche holders. Secondly, since the total perception of participation is fixed, a decrease in senior-tranche holders' perception of participation must be accompanied by an increase in junior-tranche holders' perception of the same amount. As such, an uneven allocation of perception of participation benefits junior-tranche holders more than it harms senior-tranche holders, allowing the principal to reduce the aggregate security offering. In other words, this differentiation creates a beneficial *assortative matching* within a tranching structure. To sum up, the principal should use a multi-tranche structure to differentiate agents such that senior-tranche holders participate more aggressively, which alleviates junior-tranche holders' fear of miscoordination.

Our characterization of the optimal security design provides several new insights into contract design in multi-agent settings. First, offering multiple tranches instead of a single one can be motivated purely by coordination frictions. As a comparison, in a scenario where agents can directly observe the state and coordinate perfectly under the principal's recommendation, offering all agents an identical security is optimal. In the presence of coordination frictions, the optimal security design differentiates homogeneous agents to achieve assortative matching, rather than catering to heterogeneous agents. Therefore, we justify the use of multiple tranches without resorting to any clientele effect.

Second, the desirability of differentiating agents depends on whether it allows for assortative matching between perception of participation and payoff sensitivity. Differentiation is desirable in the model because the principal can use flexible security formats to control the distribution of the two objects among agents. However, when security formats are restricted, for example, when the principal can differentiate agents only in security pricing but not in security formats, differentiation is not desirable because the restriction prevents the beneficial "assortative matching." This point contrasts our paper with existing literature on contracting with externalities (Segal, 2003; Winter, 2004; Halac et al., 2020), where the principal always prefers to differentiate agents irrespective of their payoff structures. To the best of our knowledge, our paper is the first in the literature to highlight the role of differentiating agents' payoff structures in contracting with externalities.

Third, the optimal security design induces differentiation in agents' perception of participation but may not take it to extremes. That means, under the optimal design, marginal agents may still face substantial uncertainty regarding the decisions of agents holding a different tranche. This point manifests that differentiation in our model is not intended to eliminate strategic uncertainty as in the literature of contracting with externalities.

The rest of this paper is organized as follows. Section 1 reviews the related literature. Section 2 illustrates the main insight of our results with a simplified example. Section 3 sets up the formal model. Section 4 conducts the equilibrium analysis for the sub-game of participation for any given security design. Section 5 studies the optimal security design based on the equilibrium analysis in Section 4. Section 6 is devoted to discussions and extensions. All proofs are relegated to the Appendix unless otherwise specified.

Related literature

Our paper is closely related to the large literature of the global games (Carlsson and van Damme, 1993; Morris and Shin, 1998, 2004; Frankel et al., 2003). Unlike most of the existing literature, we focus on multi-agent security design that shapes the global game to be played. The work that is closest to ours is Sákovics and Steiner (2012), who study a principal's optimal subsidies that attain a given likelihood of successful coordination at minimal cost in a coordination game with heterogeneous agents. Their exercise can be interpreted in our framework as holding agents' payoff sensitivity fixed and altering their perception of participation through subsidies. Our paper differs from theirs in two important ways. First, we start with homogeneous agents and allow differentiation of agents to emerge endogenously in equilibrium. In our setup, the principal's optimal design intentionally differentiates homogeneous agents to mitigate the adverse impact of miscoordination, rather than merely responding to their exogenous heterogeneity. Second, we address optimal security design in a general framework, where the principal can flexibly design securities subject to the budget constraint and potentially offer different securities to different agents. This flexibility leads to sharp predictions regarding the optimal security formats and allows the discussion on whether and how agents should be differentiated.

Our paper fits within the literature on contracting with externalities by examining contract design in coordination games. Prior studies by Segal (2003), Winter (2004), and Halac et al. (2020)

capture strategic risk by focusing the worst equilibrium of a coordination game with perfect information and show that extreme differentiation of agents' payoffs is always preferred to eliminate strategic uncertainty. [4][5] Similarly, Halac et al. (2021) focus on the worst equilibrium but allow for rank uncertainty to create mutual assurance among agents. They find that differentiation of agents' payoff is not preferred if agents are ex ante homogeneous. In contrast, our paper introduces a state and noisy signals about the state to pin down a unique equilibrium of the coordination game, where agents provide mutual assurance through perception of participation. The principal's contract design determines both each agent's perception of participation and payoff sensitivity, and the matching between these two objects over agents is crucial in the design. Our paper's novel contribution is to show that the desirability of differentiation of agents depends on the flexibility of the payoff structures that the principal can offer to agents. This finding highlights the role of payoff structures in contract design to mitigate coordination problems, and to our knowledge, is the first study to do so. Unlike the existing literature of contracting with externalities and ours, Luo (2023) studies the implication for contracting of strategic communication in a coordination game instead of strategic risk. There whether to differentiate agents depends on whether to induce persuasion between them.

Showing the optimality of multi-tranching structures, our paper is closely related to the literature of security design. Tranching is an ubiquitous phenomenon in finance and has been rationalized from different angles in the literature. One strand of research considers security design when the seller is more informed (Leland and Pyle, 1977; Myers and Majluf, 1984; Boot and Thakor, 1993; Nachman and Noe, 1994; DeMarzo and Duffie, 1999; DeMarzo, 2005; Biais and Mariotti, 2005). DeMarzo and Duffie (1999) and Biais and Mariotti (2005) show that debt (as a single senior tranche) is optimal among monotone securities if the security is designed before the seller receives private information and chooses the level of retention. DeMarzo (2005) rationalizes the common practice of pooling and tranching. Another strand of research focuses on the case of more informed buyers (Gorton and Pennacchi, 1990; Demarzo et al., 2005; Axelson, 2007; Dang et al., 2015; Yang, 2020), which suggest that debt is the least information-sensitive security and can best protect the seller from buyers' exogenous or endogenously acquired private information. In our paper, agents privately observe signals about the state, and a tranching structure is used to reduce their information rent as well. While existing literature suggests offering a single senior tranche to all investors, Winton (1995) and Friewald et al. (2016) rationalize multiple tranches based on different

[4] To see this, consider a game with two agents. To ensure that neither agents participating is not an equilibrium, the principal should offer agent 1 a sufficiently high payoff so that he wants to participate regardless of agent 2's decision. Then the principal can offer agent 2 a lower payoff because he knows that agent 1 will surely participate. In this example, agent 1 provides assurance to agent 2 and assumes no assurance from him.

[5] The optimality of such extreme differentiation is not sensitive to the specific payoff structure, although these papers obtain it based on simple ones..

mechanisms. Winton (1995) considers multiple investors in the classical costly state verification framework and shows that multiple tranches can economize on verification cost by allowing different investors to focus on the verification of different subsets of states. Friewald et al. (2016) indicate that multiple tranches could be favorable due to a post-sale clientele effect, where buyers may have heterogeneous holding costs, and multiple tranches can cater to their varying needs for liquidity. In contrast, our paper suggests that multiple tranches be created to deliberately differentiate ex ante homogeneous agents in the presence of coordination frictions, which is the first in the literature to our knowledge and has interesting implications for security design.

2 An Illustrative Example

This section presents an example to illustrate the paper's main insights. Specifically, we adapt the debt rollover model of Morris and Shin (2004) to our study of security design.

2.1 Setup of the Example

There are three dates, $t = 0, 1, 2$. All players do not discount future cash flows. A firm (i.e., the principal) raises \$1 from each of two banks (i.e., the agents) to develop a project at $t = 0$, and the project outcome is realized at $t = 2$. At $t = 0$, the firm enters into a loan contract with each bank, which specifies that Bank i lends \$1 to the firm at $t = 0$ in exchange for a debt payment at $t = 2$. At $t = 1$, each bank receives private information and can decide whether to terminate the loan contract or not. Termination means forgoing the debt payment and getting the \$1 investment back from the firm immediately. We say a bank participates if he decides not to terminate.

At $t = 2$, the project either succeeds or fails. The firm value will be $C > 0$ if the project fails and sufficiently high if it succeeds. The debt payment to Bank i is state-contingent and specified by the debt security (c_i, d_i), where c_i is the payment upon failure and d_i the payment upon success, respectively. We refer to $d_i - c_i$ as Bank i's *payoff sensitivity*. The debt security (c_i, d_i) can be interpreted as follows. When the project fails, the firm is worth only the liquidation value of its tangible assets, which is C, and Bank i only receives the tangible assets assigned to him as collateral, which is c_i. When the project succeeds, the firm value is so high relative to its liquidation value C that the firm is willing to pay the debt face value d_i to avoid forced liquidation. Due to the resource constraint, the total amount of collateral cannot exceed the total value of tangible assets, i.e., $c_1 + c_2 \leq C$. We assume $C < 1$, i.e., the firm does not have sufficient tangible assets to fully collateralize any loan.

The project's success probability depends on an exogenous state $\theta \in \mathbb{R}$ and the number of participating banks . In particular, in state θ, the success probability equals $P_1(\theta)$ if both banks participate and $P_0(\theta)$ otherwise. Both $P_0(\theta)$ and $P_1(\theta)$ are continuous and increasing in θ.We assume $P_1(\theta) > P_0(\theta)$. That is, the project is more likely to succeed when more banks participate. This assumption implies complementarity between the banks' participation decisions. We also assume the existence of dominance regions so that we can apply the global game approach:$P_0(\theta)$ and $P_1(\theta)$ go to 1 as $\theta \to +\infty$ and 0 as $\theta \to -\infty$.

The firm and the banks share a common prior of θ at $t = 0$. To simplify the illustration, we assume that the common prior is the improper distribution over $\mathbb{R}$. At $t = 1$, before deciding whether to participate, each Bank i observes hisa private signal $x_i = \theta + \sigma\varepsilon_i$, where noise ε_i follows cumulative distribution function $\Phi(\cdot)$ (with probability density function $\phi(\cdot)$) and is independent of the state θ and noise ε_j for $j \neq i$, and σ is the magnitude of the noise. Note that when $\sigma = 0$, the banks can make decisions fully contingent on θ and the incomplete information participation game at $t = 1$ reduces to a continuum of complete information participation games indexed by θ. To focus on the implications of coordination frictions for security design, we perturb these complete information games by letting σ be strictly positive but close to zero. This allows us to highlight strategic uncertainty to sharpen the insight and the intuition of our results.

2.2 Banks' Participation Decisions

Since the banks' initial investment is refundable at $t = 1$ and they do not discount the cash flows, they always enter into the loan agreements at $t = 0$. We thus only need to consider the banks' participation decisions at $t = 1$. As is well understood in the global games literature, for sufficiently small σ, the game has a unique equilibrium in which the two banks play switching strategies. That is, Bank i participates if and only if his private signal $x_i > \hat{x}_i$ for some cutoff $\hat{x}_i$.[6] We say that Bank i is marginal if he observes a signal equal to his cutoff $\hat{x}_i$. When Bank i is marginal, as an equilibrium condition, he must be indifferent to participation or termination, i.e.,

$$c_i + (d_i - c_i)p_i = 1, \tag{1}$$

where

$$p_i = \int_{-\infty}^{\infty} \left\{ P_0(\theta) + [P_1(\theta) - P_0(\theta)] \left[1 - \Phi\left(\frac{\hat{x}_j - \theta}{\sigma}\right) \right] \right\} \frac{1}{\sigma}\phi\left(\frac{\hat{x}_i - \theta}{\sigma}\right) d\theta \tag{2}$$

represents the project's success probability perceived by the marginal Bank i conditional on his participation. In particular, $\frac{1}{\sigma}\phi\left(\frac{\hat{x}_i-\theta}{\sigma}\right)$ is the probability density of his posterior belief of θ, and $1 - \Phi\left(\frac{\hat{x}_j-\theta}{\sigma}\right)$ is the probability that Bank j participates when the state is θ.

[6]Here we assume that Bank i does not participate if he observes $x_i = \hat{x}_i$. This tie-breaking rule is without loss of generality, as observing $x_i = \hat{x}_i$ is a zero-probability event.

For small σ, p_i has an intuitive expression

$$p_i = P_0(\hat{x}_i) + f\left(\frac{\hat{x}_j - \hat{x}_i}{\sigma}\right) \cdot [P_1(\hat{x}_i) - P_0(\hat{x}_i)] + O(\sigma), \tag{3}$$

where

$$\begin{aligned} f\left(\frac{\hat{x}_j - \hat{x}_i}{\sigma}\right) &= \int_{\theta=-\infty}^{+\infty} \left[1 - \Phi\left(\frac{\hat{x}_j - \theta}{\sigma}\right)\right] d\left[1 - \Phi\left(\frac{\hat{x}_i - \theta}{\sigma}\right)\right] \\ &= 1 - \int_0^1 \Phi\left(\Phi^{-1}(1-y) + \frac{\hat{x}_j - \hat{x}_i}{\sigma}\right) dy. \end{aligned} \tag{4}$$

From the marginal Bank i's perspective, θ is almost equal to $\hat{x}_i$. If Bank j participates (terminates), the project's success probability is almost $P_1(\hat{x}_i)$ ($P_0(\hat{x}_i)$). In particular, $f\left(\frac{\hat{x}_j - \hat{x}_i}{\sigma}\right)$ represents the probability that Bank j participates from the marginal Bank i's perspective. Hence, Equation (3) reads that, from the marginal Bank i's perspective, the project's success probability equals the success probability without Bank j's participation $P_0(\hat{x}_i)$ plus the participation probability of Bank j (i.e., $f\left(\frac{\hat{x}_j - \hat{x}_i}{\sigma}\right)$) multiplied by the resultant increase in success probability (i.e., $P_1(\hat{x}_i) - P_0(\hat{x}_i)$).

We refer to $f\left(\frac{\hat{x}_j - \hat{x}_i}{\sigma}\right)$ as Bank i's *perception of participation*, which is a function of $\Delta_{ij} \equiv \left(\hat{x}_j - \hat{x}_i\right)/\sigma$, the relative distance between Bank j and i's cutoffs. Note that by definition, $\Delta_{ij} = -\Delta_{ji}$ and $f(\Delta) \in [0,1]$ for any $\Delta \in R$. The following lemma presents two important properties of this function.

Lemma 1. *The following two properties of $f(\cdot)$ always hold.*

- *Bank i's perception of participation is strictly decreasing in Δ_{ij}.*
- *The sum of the two banks' perception of participation is equal to 1.*

The first property immediately follows

$$\frac{df(\Delta)}{d\Delta} = -\int_0^1 \phi\left(\Phi^{-1}(1-y) + \Delta\right) dy < 0. \tag{5}$$

Intuitively, the higher Bank j's cutoff is, the less likely that Bank j will participate from a marginal Bank i's perspective. The second property follows

$$f(\Delta_{12}) + f(\Delta_{21}) = 1. \tag{6}$$

and is implied by the Bayes' rule under the improper prior.[7] Note that this property follows the same logic of the usual Laplacian belief in the global game models and nests it as a special case with homogeneous players. In that case, the two banks' cutoffs coincide, i.e., $\Delta_{12} = 0$, so that upon observing the cutoff signals, each bank believes that the other bank participates with probability 0.5. Like in Sákovics and Steiner (2012), this property serves as a budget constraint for security design.

2.3 Benefits from Differentiation

We show that the firm is strictly better off by offering banks different securities than identical ones. We compare two sets of securities. The benchmark one is the identical securities (c, d) to both banks. Then they will choose the same cutoff, say $\hat{x}$. Since debt cannot be fully collateralized, we have $c < 1 < d$. The alternative one is $(c_1, d_1) = (c + \alpha, d_1)$ and $(c_2, d_2) = (c - \alpha, d_2)$, where the face values d_1 and d_2 are chosen such that $\hat{x}_1 = \hat{x} - \Delta\sigma$ and $\hat{x}_2 = \hat{x}$.

Two points regarding the benchmark and the alternative are in order. First, for small σ, they bring the firm almost the same amount of capital in all states because $\hat{x}_1$ and $\hat{x}_2$ are almost equal to $\hat{x}$. Second, they take up the same amount of total collateral, which is $2c$. Hence, the firm essentially prefers the one with the lower total face value. In the rest of Section Section 2, we fix c and $\hat{x}$, and index the alternative set of debt securities by (α, Δ).

Proposition 1. *There exists strictly positive α and Δ, such that the corresponding total face value $d_1 + d_2$ is strictly lower than $2d$.*

This proposition confirms that rather than offer the two banks identical debt securities, the firm would strictly prefer to assign more collateral to one bank so that this bank has a slightly lower cutoff in equilibrium and thus lower *perception of participation* upon observing his cutoff. We next explain why and how this differentiation reduces the total face value.

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[7]

$$
\begin{aligned}
& f(\Delta_{12}) + f(\Delta_{21}) \\
= & \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{\hat{x}_2 - \theta}{\sigma}\right)\right] d\left[1 - \Phi\left(\frac{\hat{x}_1 - \theta}{\sigma}\right)\right] + \int_{-\infty}^{\infty} \left[1 - \Phi\left(\frac{\hat{x}_1 - \theta}{\sigma}\right)\right] d\left[1 - \Phi\left(\frac{\hat{x}_2 - \theta}{\sigma}\right)\right] \\
= & \int_{-\infty}^{\infty} d\left[1 - \Phi\left(\frac{\hat{x}_2 - \theta}{\sigma}\right)\right]\left[1 - \Phi\left(\frac{\hat{x}_1 - \theta}{\sigma}\right)\right] = 1.
\end{aligned}
$$

2.4 Intuitive Derivation

To understand how the total face value $d_1 + d_2$ varies with (α, Δ), plugging $(c+\alpha, d_1)$ and $(c-\alpha, d_2)$ into Equation (1) and Equation (3), i.e.,

$$\begin{cases} c+\alpha+(d_1-c-\alpha)p_1(\Delta)=1 \\ c-\alpha+(d_2-c+\alpha)p_2(\Delta)=1 \end{cases}, \tag{7}$$

where

$$\begin{cases} p_1(\Delta)=P_0(\hat{x})+[P_1(\hat{x})-P_0(\hat{x})]\cdot f(\Delta)+O(\sigma) \\ p_2(\Delta)=P_0(\hat{x})+[P_1(\hat{x})-P_0(\hat{x})]\cdot f(-\Delta)+O(\sigma) \end{cases}. \tag{8}$$

The firm's objective is to reduce $d_1 + d_2$ subject to the two constraints in Equation (7).

Combining the two indifference conditions in Equation (7) with Equation (8) plugged in, we obtain a combined indifference condition as follows.

$$\begin{aligned} 2 = & \underbrace{2c+[P_0(\hat{x})+[P_1(\hat{x})-P_0(\hat{x})]\cdot f(0)]\cdot(d_1+d_2-2c)}_{\text{the banks' aggregate expected payoff without differentiation}} \\ & + \underbrace{[P_1(\hat{x})-P_0(\hat{x})]\cdot[(d_1-c-\alpha)\cdot(f(\Delta)-f(0))+(d_2-c+\alpha)\cdot(f(-\Delta)-f(0))]}_{\text{the banks' additional aggregate expected payoff due to differentiation}} + O(\sigma). \end{aligned} \tag{9}$$

The left-hand side of Equation (9) represents the sum of the two banks' expected payoff upon observing their respective cutoffs, which is 2. The first term in the right-hand side is the two banks' aggregate expected payoff if they have the same perception of participation (i.e., $f(0)$), which is determined by the aggregate debt $(2c, d_1 + d_2)$. The second term refers to their additional aggregate expected payoff due to the differentiation in their perception of participation.

Fix an $\alpha > 0$ and regard d_1 and d_2 as functions of Δ. Consider a small deviation of Δ from 0. The first order effect of adjusting Δ on the banks' additional aggregate expected payoff due to differentiation is captured by

$$(d_1(0)-c-\alpha)\cdot f(\Delta)+(d_2(0)-c+\alpha)\cdot f(-\Delta), \tag{10}$$

which is the sum of the product of *payoff sensitivity* and *perception of participation* over the two banks.[8] By Equation (9), reducing the total face value $d_1 + d_2$ amounts to increasing this sum of products subject to Equation (6), the "budget constraint" on the banks' *perception of participation*. Hence, the firm should allocate more (less) *perception of participation* to the bank with higher (lower) *payoff sensitivity*. When $\Delta = 0$, the two banks have exactly the same cutoff. Since $\alpha > 0$, Bank 1 receives more collateral than Bank 2 and thus due to the indifference condition, must have a lower face value, i.e., $d_1(0) < d_2(0)$. This implies that Bank 1 has a lower *payoff sensitivity* , i.e.,

$$d_1(0) - (c + \alpha) < d_2(0) - (c - \alpha). \tag{11}$$

Recall that $f(\Delta)$ is decreasing in Δ. Therefore, a positive pair (α, Δ) is desirable to the firm, as it differentiates the banks in both *payoff sensitivity* and *perception of participation* and meanwhile induces a desirable "*assortative matching*" between the two objects.

Moreover, the optimal debt securities should concentrate all collateral on one bank, i.e., $c_1 = C$ and $c_2 = 0$. Two observations lead to this result. First, when $\Delta = 0$, the second term in the right-hand side of Equation (9) vanishes and the aggregate constraint implies that $d_1(0) + d_2(0)$ does not depend on α. This is because when the two banks have a common cutoff, they perceive the same success probability upon observing their cutoff signals. Second, concentrating all collateral on one bank polarizes the two banks' *payoff sensitivities* (see Equation (11)), maximizing the (desirable) impact of differentiation in *perception of participation* (i.e., increasing Δ from 0) on Equation (10). Concentrating all collateral on one bank, say, Bank 1, implies a tranching structure in which the senior tranche holder (i.e., Bank 1) has both the lower *payoff sensitivity* and the lower *perception of participation*. We will show that this pattern of *assortative matching* and the resultant tranching structure remain optimal in our general setup with multiple investors and general production technology.

2.5 Suboptimality of Differentiation under Collinearity Constraint

Although differentiation is crucial to the assortative matching mechanism, it is not beneficial per se. To illustrate this point, we examine a case with practical relevance. In some settings, for example syndicated loans, the firm usually offers identical securities to banks and can differentiate them only through upfront fees. This amounts to imposing a collinearity constraint on banks' payoffs, i.e., $c_2/c_1 = d_2/d_1$. If banks are differentiated, without loss of generality, suppose $c_1 > c_2$. On

[8] The banks' additional aggregate expected payoff due to differentiation is proportional positively to

$$\begin{aligned}&(d_1(\Delta) - c - \alpha) \cdot (f(\Delta) - f(0)) + (d_2(\Delta) - c + \alpha) \cdot (f(-\Delta) - f(0)) \\ =&(d_1(0) - c - \alpha) \cdot f(\Delta) + (d_2(0) - c + \alpha) \cdot f(-\Delta) - f(0) \cdot (d_1(0) + d_2(0) - 2c) + o(\Delta)\end{aligned}$$

up to a scale factor $P_1(\hat{x}) - P_0(\hat{x})$, which is a strictly positive constant invariant to the choice of (α, Δ).

the one hand, Bank 1 has an unambiguously better offer than Bank 2, so Bank 1 has the lower cutoff and thus the lower *perception of participation*. On the other hand, the collinearity constraint implies $d_1 - c_1 > d_2 - c_2$, so Bank 1 also has a higher *payoff sensitivity* than Bank 2. In this case, differentiation always leads to *negative assortative matching* between *payoff sensitivity* and *perception of participation,* which is undesirable.

This case manifests that whether differentiation is preferable depends on whether agents' payoff structures can be differentiated in certain ways. This point helps contrast our paper to the existing literature of contracting with externalities (Segal, 2003; Winter, 2004; Halac et al., 2020), where differentiation is preferable irrespective of agents' payoff structures. In these models, the principal designs contracts that determine the payoffs of a complete information coordination game played by the agents, which naturally admits multiple equilibria. To capture the "strategic risk" in a complete information game, these models require the principal to evaluate a contract according to the worst equilibrium outcome of the resultant game. This combination of equilibrium selection and the complete information setting prohibits agents from providing assurance in a mutual way,[9] so that extreme differentiation of agents' payoffs to eliminate the "strategic risk" is always preferable. In contrast, the global game approach allows fine-tuning of strategic risk perceived by agents, which is characterized by $f(\Delta)$ and can be fine tuned by the security design through adjusting Δ. Whether differentiation is preferable depends on how it affects the matching between *payoff sensitivity* and *perception of participation*.

3 The Formal Model

This section sets up the formal model with multiple agents and general production technologies. We formalize it as a joint task problem with one principal (she) and a continuum of agents (he) of unit mass indexed by $i \in [0,1]$.[10] The principal has a project that requires the agents' participation. She offers a (potentially different) security to each agent who then decides whether to participate in the project. The output of the project depends (stochastically) on the mass of participating agents as well as an exogenous fundamental state. Each agent's opportunity cost of participation is normalized to 1. Both the principal and the agents are risk-neutral and do not discount the future cash flows. In the context of our illustrative example, the firm and the banks are the principal and the agents, respectively. Accordingly, the mass of the participating agents can be interpreted as the amount of capital invested in the project. We show that our qualitative results and the assortative matching mechanism highlighted in the illustrative example remain valid in this general setup.

[9] Halac et al. (2021) allow mutual assurance among agents by introducing incomplete information on agents' payoffs.

[10] We will use she/her to refer to the principal and he/his to the agents throughout the article. We do not intentionally associate the players with particular genders.

3.1 The Production Technology

Again, let $\theta \in \mathbb{R}$ denote the exogenous fundamental state. The principal and the agents share a common prior of θ, characterized by a cumulative distribution function (c.d.f.) $H(\theta)$ and the corresponding probability density function (p.d.f.) $h(\theta)$. We assumes that $h(\theta)$ is continuous, bounded, and fully supported on $(-\infty,+\infty)$. Given the value of θ and the mass of participating agents K, the project value z follows a c.d.f. $G(z;\theta,K)$, which has a p.d.f. $g(z;\theta,K)$. We assume that the project value z is non-negative. We make the following assumptions about $g(z;\theta,K)$.

Assumption 1.

1. *The expected project value conditional on θ and K is finite, i.e.,*

$$E[z|\theta,K] \equiv \int_0^{+\infty} zg(z;\theta,K)\,dz < +\infty.$$

2. *The expected project value conditional on K is finite, i.e.,*

$$E[z|K] \equiv \int_{-\infty}^{\infty} E[z|\theta,K]\,h(\theta)d\theta < +\infty.$$

3. *If $(\theta_1,K_1) \geq (\theta_2,K_2)$ and $(\theta_1,K_1) \neq (\theta_2,K_2)$, then $g(z;\theta_1,K_1)/g(z;\theta_2,K_2)$ is strictly increasing in z.*
4. *As a function of z, the p.d.f. $g(z;\theta,K)$ has full support over $[0,+\infty)$.*

The first two conditions of Assumption 1 ensure that the principal-agent problem is well defined. The third condition states that the p.d.f. $g(z;\theta,K)$ satisfies the strict monotone likelihood ratio property (SMLRP). The higher the fundamental state and the participation of agents, the more likely the project will have a higher value. As an immediate implication of this assumption, the expected project value is strictly increasing in θ and K, i.e.,

$$E[z|\theta_1,K_1] > E[z|\theta_2,K_2]$$

if $(\theta_1,K_1) \geq (\theta_2,K_2)$ and $(\theta_1,K_1) \neq (\theta_2,K_2)$.

The fourth condition means that the c.d.f. $G(z;\theta,K)$ is strictly increasing in z and $G(0;\theta,K) = 0$. Hence, the project value is positive almost surely. It is consistent that in practices, there are always some assets left, even if a project fails or a firm goes to bankruptcy.

To streamline the analysis, we assume that $g(z;\theta,K)$ takes the form of generalized regime change, i.e.,

$$g(z;\theta,K) = \begin{cases} g_0(z;\theta,K), & K < \lambda(\theta) \\ g_1(z;\theta,K), & K \geq \lambda(\theta) \end{cases},$$

where $\lambda(\theta)$ is weakly decreasing in θ and $g_0(z;\theta,K)$ and $g_1(z;\theta,K)$ are Lipschitz continuous as follows.[11]

Assumption 2. *Let $E_i[\cdot|\theta,K]$ denote the expectation under p.d.f. $g_i(z;\theta,K)$ for $i=0,1$. There exist an $R>0$, such that*

$$|E_i[z|\theta,K_1]-E_i[z|\theta,K_2]|\leq|K_1-K_2|$$

and

$$|E_i[z|\theta_1,K]-E_i[z|\theta_2,K]|<R|\theta_1-\theta_2|$$

for all θ_1, θ_2, K_1 and K_2.

The Lipschitz continuity implies that a group of agents' participation does not generate positive surplus unless it triggers regime change. This assumption helps manifest the implication of the coordination friction for security design. It is worth noting that our main results regarding security design do not require this assumption. In Section 6.3, we extend the model to show that our main results still hold under more general production technologies. The complication there is that the principal may want to differentiate agents due to the production technology per se, but the coordination friction gives her further incentive to do so.

3.2 Contracting

The principal can choose a subset of agents and make a take-it-or-leave-it offer to each of them. An offer is a security that specifies the payment to an agent if he participates. As standard in the literature of contracting with externalities (Segal, 2003; Winter, 2004; Halac et al., 2020), we assume that the principal can rely on only bilateral contracts. That is, the payment to an agent does not depend on other agents' participation decisions except insofar as those decisions affect the project value. We motivate this assumption for two reasons. First, in financial contracting, the asset pledged to an investor is often the collateral assigned to him. It is unusual to make the amount of collateral contingent on other investors' decisions. Second, it is often difficult to verify others' participation in practice. We further assume that the project value z is verifiable but the fundamental state θ is not. Hence, the contract offered to each agent i is a security $s^i[z]$ backed by the project value z, and the sum of the security payments to all agents should not exceed the project value. Let I denote the set of agents who receive offers. Each agent in I either participates in the project at the opportunity cost of 1 or rejects. Throughout the paper, unless otherwise specified, "the agents" refer to those in I.[12] The principal's payoff equals the project value minus the total

[11] In Subsection 6.3, we show that our main results hold with more general production technology.

[12] It is without loss of generality to allow the principal to select a subset of agents. If the principal is required to make offers to all the agents, she can always offer some of them undesirable securities that will be declined for sure (e.g., securities with zero payment).

payment to the participating agents.

We assume that the principal can offer no more than $N \geq 1$ types of securities. [13] Suppose that the k-th type security $s_k[\cdot]$ is offered to a measure Q_k of agents. We refer to these agents as Type-k agents. Define $S_k[\cdot] \equiv Q_k s_k[\cdot]$ to represent the aggregate security offered to Type-k agents. Then, $\{(S_k, Q_k)\}_{k=1}^n$ represents the security bundle designed by the principal.[14]. We impose the following feasibility constraints on the security design.

Assumption 3. *The securities should satisfy the following conditions:*

1. *(Budget Constraint)* $\sum_{k=1}^n S_k[z] \leq z$.
2. *(Nonnegativity and Monotonicity) For any Type-k,* $S_k[z]$ *is nonnegative and weakly increasing in z.*
3. *(Lipschitz Continuity) For any Type-k,* $S_k[z]$ *is Lipschitz continuous in z.*
4. *(Nontriviality) For any Type-k, there exists a* $z_k > 0$, *such that* $s_k[z] > 1$ *for all* $z > z_k$.

The first condition requires that the total payment to agents does not exceed the project value. This budget constraint ensures that the principal can always fulfill her financial commitment to the agents. The second condition requires the securities to be nonnegative, since the agents are protected by the limited liability. It also imposes the monotonicity constraint on the securities, which is a usual assumption in the literature and well micro-founded by the "side loan" argument as in Innes (1990). The monotonicity constraint together with the SMLRP of $g(z;\theta,K)$ implies the strict monotonicity of the expected payment in θ and K, i.e.,

$$E[S_k[z]|\theta_1, K_1] > E[S_k[z]|\theta_2, K_2]$$

if $(\theta_1, K_1) \geq (\theta_2, K_2)$ and $(\theta_1, K_1) \neq (\theta_2, K_2)$.

The third condition imposes the Lipschitz continuity of the security payment in the project value. The Lipschitz continuity can be implied by the dual monotonicity constraint usually assumed in the security design literature and thus is a weaker condition.[15] The fourth condition requires that the security payment $s_k[z]$ should not be uniformly below 1, the opportunity cost of the agents. Otherwise, the Type-k agents will always reject the offer and thus it is equivalent to removing these agents from I.

Finally, define

$$T_F[z] = \min\{z, F\}$$

[13] Although N is fixed as a parameter, it is allowed to take any positive integer value.

[14] Since agents are homogeneous when the security bundle is offered, how different securities are allocated among agents does not matter to the principal. She can offer each agent a lottery, the outcome of which specifies the security for that agent.

[15] The dual monotonicity constraint requires that the residual cash flow received by the principal is also weakly increasing, i.e., $z - \sum_{k=1}^n S_k[z]$ is weakly increasing in z.

i.e., a senior tranche with face value F. For $0 < F_1 < F_2$, we call $T_{F_2} - T_{F_1}$ a junior tranche relative to T_{F_1}.

Remark 1. It is without loss of generality to assume that the principal offers only one security to an agent in our setup. In Section 6.4, we show that if the principal is restricted to bilateral contracts and subject to the budget constraint, it is suboptimal for her to offer more than one security to an agent.

3.3 The Timeline and Information

The model has three dates. At date 0, the principal offers a security bundle to a subset of agents.

At date 1, nature draws a value of θ, and each agent i observes a noisy signal of θ, $x^i = \theta + \sigma\varepsilon^i$. As in Section 2, ε_i follows cumulative distribution function $\Phi(\cdot)$ (with probability density function $\phi(\cdot)$) and is independent of the state θ and noise ε_j for $j \neq i$, and σ captures the magnitude of the noise. Upon observing his signal x^i, agent i decides whether to participate in the project.

At date 2, the project value is realized. Participating agents receive their respective security payments, and the principal receives the residual value of the project.

We assume that p.d.f. ϕ is continuous and fully supported on $(-\infty, +\infty)$.[16] We also require ϕ to satisfy the strict monotone likelihood ratio property (SMLRP), i.e., for any $a > 0$, $\phi(\varepsilon)/\phi(\varepsilon + a)$ is strictly increasing in ε.

3.4 Security Design under Coordination Frictions

Agents' decisions are strategic complements in this game because each agent's security payment weakly increases in the project value, which in turn strictly increases in other agents' participation. As a result, an agent's payoff of participating weakly increases in the participation of others. This creates motives for an agent to coordinate his decision with others at date 1. However, since agents only observe noisy signals, they know neither the state nor the mass of participating agents exactly. The subgame at date 1 is thus a coordination game with incomplete information, which is essentially a global game with agents' (potentially heterogeneous) payoffs designed by the principal at date 0. As is standard in the global game literature (Frankel et al., 2003), we introduce the following technical assumption about the production technology to ensure there exists a unique equilibrium of the subgame.

Assumption 4. *$lim_{\theta \to -\infty} E\left[z|\theta, 1\right] = 0$; and for any $\hat{z} > 0$, $lim_{\theta \to \infty} G(\hat{z}; \theta, 0) = 0$.*

[16] We make this full support assumption for technical convenience. Under certain conditions, our main results hold for bounded support as well.

This assumption requires that the project's expected value vanishes when the fundamental state is sufficiently weak, even though all agents participate; and that the project's value can be arbitrarily large when the fundamental state is sufficiently strong, even though no agent participates. As an implication, for any given finite-type security bundle, the subgame at date 1 has dominance regions and hence the contagion argument of the global game approach works. Lemma Lemma 2 below formalizes this implication.

Lemma 2. *There exists a $\underline{\theta}$ and $\overline{\theta}$ such that for all $K \geq 0$, $E[s_k[z]|\theta,K] - 1 < 0$ for $\theta \leq \underline{\theta}$ and $E[s_k[z]|\theta,K] - 1 > 0$ for $\theta \geq \overline{\theta}$.*

Assumption 4 is not a stringent one, as it allows $E[z|\theta,1]$ and $G(\hat{z};\theta,0)$ to go to 0 arbitrarily slowly. It is satisfied by many popular probability distributions on $\mathbb{R}^+$, such as the lognormal distributions $\ln z \sim N(\theta + K, \sigma^2)$ and the exponential distribution with rate parameter $e^{-(\theta+K)}$. It nicely ensures the uniqueness of the equilibrium but otherwise does not affect our results qualitatively.

As standard in the global game literature, we study the case in which σ is strictly positive but close to zero. This allows us to focus on the strategic uncertainty, which gives rise to the miscoordination risk, rather than the fundamental uncertainty about θ. In particular, we address the following two questions. First, what security format should be used in the presence of coordination frictions? This question is central to the literature of security design. Second, should the agents be differentiated, and if so, how? This question is central to the literature of contracting with externalities. To rule out the case of trivial differentiation without generating extra benefit to the principal, we focus on the qualitative properties of optimal security bundles with the fewest types of securities.

4 The Equilibrium Following Any Security Offering

We solve the security design problem by backward induction. This section derives the unique equilibrium of the subgame at date 1 for any security bundle. The next section studies the optimal design of securities based on the equilibrium analysis in this section.

4.1 Intuitive Derivation

Suppose that a bundle of n types of securities is offered in date 0. Consider a symmetric equilibrium in which for each type k in $\{1,2,\ldots,n\}$, all Type-k agents play a switching strategy characterized by a cutoff $\hat{x}_k^\sigma$.[17] That is, each Type-k agent i participates if and only if he observes $x^i \geq \hat{x}_k^\sigma$.

[17] As is well known in the global games literature, it is without loss of generality to focus on symmetric equilibria with switching strategies when the noise is small.

Let $m_k^\sigma(\theta)$ be the probability that a Type-k agent participates if the state is θ. Then

$$m_k^\sigma(\theta) = Pr[x^i \geq \hat{x}_k^\sigma|\theta] = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right). \tag{12}$$

As is usual in models with a continuum of players, we adopt the law of large numbers convention[18] so that the mass of the participating agents is

$$M^\sigma(\theta) \equiv \sum_{k=1}^{n} Q_k m_k^\sigma(\theta) = \sum_{k=1}^{n} Q_k - \sum_{k=1}^{n} Q_k \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right), \tag{13}$$

which is strictly increasing in θ. As before, we refer to an agent as marginal if he observes his cutoff signal. A marginal Type-k agent must break even in expectation, so

$$\int_{-\infty}^{\infty}\left[\int_0^{\infty}(s_k[z]-1)\,g\left(z;\theta,M^\sigma(\theta)\right)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right)h(\theta)d\theta = 0. \tag{14}$$

It is worth noting that the posterior probability of θ contains two sources of information: one is the prior $h(\theta)$, and the other is the agent's private signal $x^i = \hat{x}_k^\sigma$.

For small σ, the private signal is sufficiently accurate relative to the prior information. We can thus simplify a marginal agent's breakeven condition to

$$\int_{-\infty}^{\infty}\left[\int_0^{\infty}(s_k[z]-1)\,g\left(z;\hat{x}_k^\sigma,M^\sigma(\theta)\right)dz\right]\frac{1}{\sigma}\phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right)d\theta = O(\sigma).$$

While the fundamental uncertainty (about θ) almost vanishes, the strategic uncertainty (regarding other agents' participation) remains substantial, because $M^\sigma(\theta)$ becomes extremely sensitive to θ around the cutoff. In particular, from a marginal Type-k agent's perspective, θ is highly possible to be in an $O(\sigma)$-neighborhood of $\hat{x}_k^\sigma$, where $m_k^\sigma(\theta)$, the probability that other Type-k agents participate, ranges from almost 0 to almost 1, making $M^\sigma(\theta)$,the mass of participating agents, vary by almost Q_k in the $O(\sigma)$-neighborhood of $\hat{x}_k^\sigma$. As a result, the marginal Type-k agent can be very uncertain about others' participation. To better represent the strategic uncertainty, we rewrite the left-hand side of the above equation as an integral with respect to the mass of participating agents, i.e.,

$$\int_0^1\left[\int_0^{\infty}(s_k[z]-1)\,g\left(z;\hat{x}_k^\sigma,M^\sigma(\theta)\right)dz\right]\frac{dm_k^\sigma(\theta)}{dM^\sigma(\theta)}dM^\sigma(\theta) = O(\sigma).$$

The strategic uncertainty faced by a marginal Type-k agent is captured by $\frac{dm_k^\sigma(\theta)}{dM^\sigma(\theta)}$, which is a

[18] The law of large numbers is not well defined for a continuum of random variables (Sun, 2006). Our convention is equivalent to assuming that the agents' play is the limit of play of finite selections from the population.

function of the mass of participating agents, $M^\sigma(\theta)$, the mass of each type j, Q_j, and the relative distance between Type-j and Type-k agents' switching cutoffs, $\frac{\hat{x}_j^\sigma - \hat{x}_k^\sigma}{\sigma}$, i.e.,

$$\frac{dm_k^\sigma(\theta)}{dM^\sigma(\theta)} \equiv f\left(M^\sigma(\theta); \left\{Q_j, \frac{\hat{x}_j^\sigma - \hat{x}_k^\sigma}{\sigma}\right\}_{j=1}^n\right).$$

Formally, we define f is as follows.[19]

Definition 1. For any type $k \in \{1, 2, \ldots, n\}$ and any $\{Q_j, \Delta_{k,j}\}_{j=1}^n$,

$$f\left(M; \{Q_j, \Delta_{k,j}\}_{j=1}^n\right) \equiv \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q_j \phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)},$$

where m_k is a function of M implicitly defined by

$$M = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right).$$

As $\sigma \to 0$, each switching cutoff $\hat{x}_j^\sigma$ converges to a real number $\hat{x}_j$, and each relative distance $(\hat{x}_j^\sigma - \hat{x}_k^\sigma)/\sigma$ converges to a real number or goes to infinity, denoted by $\Delta_{k,j} \in \mathbb{R} \cup \{+\infty, -\infty\}$. For a given set of masses $\{Q_j\}_{j=1}^n$ and relative distances $\{\Delta_{l,j}\}_{l,j=1}^n$, to simplify the notation, we define for each Type-k

$$f_k(M) = f\left(M; \{Q_j, \Delta_{k,j}\}_{j=1}^n\right), \tag{15}$$

Then the marginal agent's breakeven condition becomes

$$\int_0^\infty s_k[z] \left[\int_0^\infty g(z; \hat{x}_k, M) f_k(M) dM\right] dz = 1. \tag{BE}$$

The left-hand side is the expected payoff perceived by a marginal Type-k agent, and the right-hand side is his opportunity cost of participation. We will show that $f(\cdot)$ is the probability density function of the mass of participating agents from a marginal Type-k agent's perspective. Throughout the paper, $f_k(\cdot)$ is referred to as Type-k *perception of participation*.

[19] By $m_k^\sigma(\theta) = 1 - \Phi\left(\left(\hat{x}_k^\sigma - \theta\right)/\sigma\right)$ and $M^\sigma(\theta) \equiv \sum_{k=1}^n Q_k m_k^\sigma(\theta)$, it is straightforward to see that

$$\frac{dm_k^\sigma(\theta)}{dM^\sigma(\theta)} = \frac{dm_k^\sigma(\theta)/d\theta}{dM^\sigma(\theta)/d\theta} = \frac{\phi\left(\Phi^{-1}(1-m_k^\sigma(\theta))\right)}{\sum_{j=1}^n Q_j \phi\left(\Phi^{-1}(1-m_k^\sigma(\theta)) + \frac{\hat{x}_j^\sigma - \hat{x}_k^\sigma}{\sigma}\right)}$$

and

$$M^\sigma(\theta) = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k^\sigma(\theta)) + \frac{\hat{x}_j^\sigma - \hat{x}_k^\sigma}{\sigma}\right).$$

4.2 Characterization of the Equilibrium

For a given security bundle, the subgame at date 1 is a global game in which agents have potentially heterogeneous payoffs. The following proposition characterizes the unique equilibrium of this subgame at vanishing noise.

Proposition 2. *Given a security bundle $\{(S_k, Q_k)\}_{k=1}^n$, for each $k \in \{1,2,...,n\}$, the Type-k agents' switching cutoff $\hat{x}_k^\sigma$ converges to an $\hat{x}_k \in \mathbb{R}$ as $\sigma \to 0$. In particular, $\{\hat{x}_k\}_{k=1}^n$ satisfy the following system of n equations:*

$$\int_0^\infty s_k[z] \left[\int_0^\infty g(z; \hat{x}_k, M) f_k(M) dM \right] dz = 1, \tag{16}$$

where $f_k(M)$ is given by Equation (15),

$$\Delta_{k-1,k} \begin{cases} = +\infty, & \text{if } \hat{x}_k > \hat{x}_{k-1} \\ = -\infty, & \text{if } \hat{x}_k < \hat{x}_{k-1} \\ \in [-\infty, +\infty], & \text{if } \hat{x}_k = \hat{x}_{k-1} \end{cases}, \tag{17}$$

and

$$-\Delta_{k,j} = \Delta_{j,k} = \sum_{i=j+1}^{k} \Delta_{i-1,i}. \tag{18}$$

Conversely, if $\{\hat{x}_k\}_{k=1}^n$ and $\{\Delta_{k,j}\}_{j,k\in\{1,2,...,n\}}$ satisfy Equation (16), Equation (15), Equation (17), and Equation (18), then the Type-k agents' cutoff $\hat{x}_k^\sigma$ converges to $\hat{x}_k$ as $\sigma \to 0$.

Equation (16) is the breakeven condition of a marginal Type-k agent. In particular,

$$\int_0^\infty g(z; \hat{x}_k, M) f_k(M) dM$$

is the probability density of the project value z conditional on receiving the cutoff signal $\hat{x}_k$. It can be viewed as the pricing kernel for a marginal Type-k agent such that he prices the security at 1, which is his opportunity cost of participation. To understand Equation (17) and Equation (18), note that when $\hat{x}_k > \hat{x}_{k-1}$ ($\hat{x}_k < \hat{x}_{k-1}$), by definition, the limit relative distance $\Delta_{k-1,k} = \lim_{\sigma\to 0}(\hat{x}_k^\sigma - \hat{x}_{k-1}^\sigma)/\sigma = +\infty$ ($-\infty$). When the limit cutoffs are equal, the limit relative distance can take finite values. Notably, these conditions are not only necessary but also sufficient for $\{\hat{x}_k\}_{k=1}^n$ to be the equilibrium cutoffs. Sufficiency is important in a design problem, because it guarantees that the security bundle derived based on these conditions indeed induces the desirable outcome in equilibrium.

The intuition behind this equilibrium characterization can be appreciated in an example with a two-type bundle. Let $\hat{x}_1^\sigma$ and $\hat{x}_2^\sigma$ denote the two equilibrium switching cutoffs of the Type-1 and the Type-2 agents, respectively. If the two securities are equally attractive, both types of the agents choose the same switching cutoff $\hat{x}_1^\sigma = \hat{x}_2^\sigma$, and their respective marginal agents share the same view on θ as well as the same belief on the mass of participating agents. At vanishing σ, the same view on θ results in identical conditional p.d.f. of the project value, i.e., $g(\cdot;\hat{x}_1,M) = g(\cdot;\hat{x}_2,M)$ for any given mass of participating agents M; and the same belief on the mass of participating agents is captured by the same *perception of participation* $f_1(M) = f_2(M) = 1/(Q_1+Q_2)$, where $M \in [0,Q_1+Q_2]$. As a result, the marginal agents of both types share the same p.d.f. of the project value z, i.e., $\int_0^\infty g(z;\hat{x}_1,M) f_1(M)\, dM = \int_0^\infty g(z;\hat{x}_2,M) f_2(M)\, dM$.

Now suppose security $s_1[\cdot]$ is *slightly* more attractive than $s_2[\cdot]$, resulting in $\hat{x}_1^\sigma < \hat{x}_2^\sigma$, i.e., the Type-1 agents are more eager to participate. In this case, the limit cutoffs could remain identical, i.e., $\hat{x}_1 = \hat{x}_2$, but the limit relative distance becomes strictly positive, i.e., $\Delta_{1,2} = \lim_{\sigma\to 0}(\hat{x}_2^\sigma - \hat{x}_1^\sigma)/\sigma > 0$, resembling two different channels through which the security design may affect the agents' equilibrium behavior. The first one is the fundamental channel. Since $\hat{x}_1 = \hat{x}_2$, the marginal agents of both types share almost the same belief about the fundamental state θ. Hence, in this case, the fundamental channel does not come into effect, because for any given mass of participating agents M, the marginal agents of both types hold almost the same conditional p.d.f. of the project value z, i.e., $g(\cdot;\hat{x}_1,M) = g(\cdot;\hat{x}_2,M)$. However, since $\Delta_{1,2} > 0$, they perceive others' participation differently. This is the strategic channel via which the security design effects. In particular, the marginal Type-1 agents perceive that Type-2 agents are less likely to participate while the marginal Type-2 agents perceive the opposite about Type-1. This is captured by that the Type-1 perception of participation $f_1(\cdot)$ is leftward-tilted relative to the Type-2 perception of participation $f_2(\cdot)$, i.e., as probability densities of the mass of participating agents, $f_1(\cdot)$ is first order stochastically dominated by $f_2(\cdot)$. Since $g(\cdot;\theta,M)$ satisfies SMLRP, $\int_0^\infty g(z;\hat{x}_1,M) f_1(M)\, dM$ is also first order stochastically dominated by $\int_0^\infty g(z;\hat{x}_2,M) f_2(M)\, dM$, meaning that regarding the project value z, the marginal Type-1 agents are more pessimistic relative to the marginal Type-2 agents. The relative distance $\Delta_{1,2}$ hence reflects the degree to which the marginal Type-1 (Type-2) agents are more (less) pessimistic, and in equilibrium it varies to adjust the marginal agents of both types' beliefs about the project value z, so that they both price their respective securities at 1, their opportunity cost of participation.

When security $s_1[\cdot]$ becomes *significantly* more attractive than $s_2[\cdot]$, the limit cutoffs of the two types of agents become distinct to each other, i.e., $\hat{x}_1 < \hat{x}_2$. In this case, $\Delta_{1,2} = \lim_{\sigma\to 0}(\hat{x}_2^\sigma - \hat{x}_1^\sigma)/\sigma = \infty$ so that the potential impact of security design through the strategic channel is exhausted, in the sense that the marginal Type-2 (Type-1) agents are almost sure that Type-1 (Type-2) agents will (not) participate. Although holding quite opposite beliefs regarding the other type's participation, since security $s_1[\cdot]$ is *significantly* more attractive than $s_2[\cdot]$, the divergence in beliefs cannot close the gap between the attractiveness of the two securities. As a result, the fundamental channel has to come into effect. In particular, since $\hat{x}_1 < \hat{x}_2$, the marginal agents of Type-1's (Type-2's) belief about the fundamental state θ is strictly more pessimistic (optimistic) than that of the marginal agents of Type-2 (Type-1). Hence, the marginal agents of Type-1 (Type-2) are strictly more pessimistic (optimistic) in both the fundamental state and the other type's participation. As such, the equilibrium cutoffs always make the marginal agents of both types price their respective securities at 1.

4.3 A Closer Look at the Perception of Participation

In this subsection, we take a closer look at the perception of participation, which is characterized by Proposition 3 below. This proposition helps understand how agents perceive others' participation in equilibrium and thus provides the foundation for security design.

Without loss of generality, in the rest of the paper, we number the types of securities such that $\Delta_{k-1,k}$ is nonnegative for all k. For notational convenience, let $\Delta_{0,1} = \Delta_{n,n+1} = +\infty$ and $\hat{x}_0 = -\hat{x}_{n+1} = -\infty$ for any n-type bundle. Define $L(k) \equiv \max\{j : \Delta_{k,j} = -\infty\}$ and $U(k) \equiv \max\{j : \Delta_{k,j} < +\infty\}$. An immediate implication of this definition is that $\Delta_{i,k}$ is finite for any $L(k) < i \leq U(k)$, so $L(i) = L(k)$ and $U(i) = U(k)$.

Proposition 3. *The perception of participation has the following properties:*

1. *For any $M \in \left(0, \sum_{j=1}^{n} Q_j\right)$,*

$$\sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j \Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\, dy\right) + \Delta_{k,j}\right) = M. \tag{19}$$

   In particular, $\int_0^{\sum_{j=1}^{n} Q_j} f_k(y)\, dy = 1$.

2. *$\sum_{k=1}^{n} Q_k f_k(M) = 1$ for $M \in (0, \sum_{k=1}^{n} Q_k)$.*
3. *$f_k(M)$ is positive for $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ and 0 elsewhere.*
4. *For $L(k) < i \leq U(k)$ and $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, $\int_0^M f\left(y; \{Q_j, \Delta_{k,j}\}_{j=1}^{n}\right) dy$ is strictly increasing in $\Delta_{k,i}$.*
5. *For any $L(k) < i \leq U(k)$, if $\Delta_{k,i} = 0$, then $f_k(M) = f_i(M)$; if $\Delta_{k,i} > (<) 0$, then $f_k(M)/f_i(M)$ is strictly decreasing (increasing) in M over $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$.*

The first property follows the construction of $f_k(M)$. To see this, we can write

$$\int_{M^\sigma(\theta)=0}^{M} f_k(M^\sigma(\theta))\, dM^\sigma(\theta) = \int_{M^\sigma(\theta)=0}^{M} \frac{dm_k^\sigma(\theta)}{dM^\sigma(\theta)} dM^\sigma(\theta) + O(\sigma) = m_k^\sigma(\theta) + O(\sigma),$$

where

$$M = \sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j \Phi\left(\Phi^{-1}(1-m_k^{\sigma}(\theta)) + \Delta_{k,j}\right) + O(\sigma).$$

It also confirms that $f_k(y)$ is indeed a probability distribution.

The second property states that the aggregate perception of participation equals one everywhere, which nests Equation (6) in our illustrative example as a special case. This property is an immediate implication of the Bayes' rule with improper prior. In the limit case, the agents' private information completely dominates their prior information of the state, so they act as if the prior is improper. This is an important constraint on security design.

The third property characterizes the support of perception of participation. For $i \leq L(k)$, the marginal Type-k agents perceive that the Type-i agents participate almost surely. For $i > U(k)$, the marginal Type-k agents perceive that the Type-i agents do not participate almost surely. For other types of agents, the marginal Type-k agents are uncertain whether they are participating. Therefore, from the marginal Type-k agents' perspective, the mass of participating agents must be within $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ almost surely and may take any value in the region.

The fourth property captures how the relative distance between cutoffs affects the perception of participation. When $\Delta_{k,i}$ increases, the marginal Type-k agents perceive that the Type-i agents are less likely to participate. As a result, they perceive scenarios with low levels of participation more likely, so their perception of participation shifts toward the left. This property has an important implication for security design. By altering $\Delta_{k,i}$ between the securities, the principal can adjust the marginal agents' perception of participation as well as how much they care about their security payments in scenarios with different levels of participation.

The fifth property concerns the relative perception of participation between different types. If $\Delta_{k,i} > 0$, the marginal Type-i agents are more pessimistic about others' participation than the marginal Type-k agents.

5 Security Design

This section studies the principal's security design problem. Suppose the security bundle $\{(S_k, Q_k)\}_{k=1}^n$ is offered and results in equilibrium cutoffs $\{\hat{x}_k^{\sigma}\}_{k=1}^n$. In state θ, mass $Q_k m_k^{\sigma}(\theta)$ of Type-k agents participate, so the principal's expected payoff is

$$E[\pi^P] = \int_{-\infty}^{\infty} \left[\int_0^{+\infty} \left(z - \sum_{k=1}^{n} m_k^{\sigma}(\theta) Q_k s_k[z] \right) g\left(z; \theta, M^{\sigma}(\theta)\right) dz \right] h(\theta) d\theta.$$

As σ vanishes, $m_k^\sigma(\theta)$ converges to $1\{\theta > \hat{x}_k\}$. That means, in the limit case, Type-k agents almost all participate if $\theta > \hat{x}_k$ and almost all reject if $\theta < \hat{x}_k$. Then we obtain the following Proposition 4.

Proposition 4. *As $\sigma \to 0$, the mass of participating agents in state θ converges in probability to*

$$\sum_{k=1}^{n} Q_k \cdot 1\{\theta > \hat{x}_k\},$$

and the principal's expected payoff converges to

$$\sum_{k=0}^{n} \int_{\hat{x}_k}^{\hat{x}_{k+1}} \left(E\left[z \mid \theta, \sum_{j=1}^{k} Q_j \right] - E\left[\sum_{j=1}^{k} S_j[z] \middle| \theta, \sum_{j=1}^{k} Q_j \right] \right) h(\theta) d\theta, \tag{20}$$

where $\hat{x}_0$ and $\hat{x}_{n+1}$ are defined as $-\infty$ and ∞, respectively

Note that by Assumptin 2, for $i = 0, 1$,

$$|E_i[z|\theta, K_1] - E_i[z|\theta, K_2]| \leq |K_1 - K_2|.$$

That means, agents' participation does not generate positive surplus unless it contributes to the regime change. An implication of this property is that the optimal security bundle should induce all types of agents to have identical limit cutoffs at vanishing noise. To see the intuition, suppose regime change takes place at $\hat{\theta}$. For the types of agents whose cutoffs are greater than $\hat{\theta}$, their participation has almost no impact on whether regime change takes place. When they participate, the principal must pay each of them at least 1 in expectation but the increase in the expected value of the project is smaller. The principal would be better off not offering them securities. For the types of agents whose cutoffs are smaller than $\hat{\theta}$, they may participate at $\theta < \hat{\theta}$. In these states, their participation has almost no impact on whether regime change takes place either. The principal would be better off offering them less attractive securities such that their cutoffs are $\hat{\theta}$.

Proposition 5. *The optimal security bundle must induce an identical limit cutoff $\hat{\theta}$ for all agents..*

The rest of the section is devoted to deriving the properties of optimal security bundles. By Proposition 5, we focus on security bundles that induce an identical limit cutoff $\hat{\theta}$ for all agents. Intuitively, a finer differentiation of agents always makes the principal weakly better off. With more types allowed, the principal can at least replicate any security bundle with fewer types. In addition to the trivial weak dominance, we find that proper differentiation of agents can make the principal strictly better off. To manifest the nontrivial strict dominance, we focus our attention on the optimal security bundles with the fewest types.

5.1 Directions for security design

Before formally establishing the results, we discuss the forces shaping security design. Suppose that the optimal security bundle $\{(S_k, Q_k)\}_{k=1}^{n}$ induces a common cutoff $\hat{\theta}$. Note that given the total mass of agents $K \equiv \sum_{k=1}^{n} Q_k$ and the common cutoff $\hat{\theta}$, the expected value of the project is a fixed value

$$\int_{-\infty}^{\hat{\theta}} E[z|\theta,0]\, h(\theta)d\theta + \int_{\hat{\theta}}^{+\infty} E[z|\theta,K]\, h(\theta)d\theta,$$

and the expected payment to the agents is

$$\int_{\hat{\theta}}^{+\infty} E\left[\sum_{k=1}^{n} S_k[z] \mid \theta, K\right] h(\theta)d\theta,$$

which is completely determined by the aggregate security. Essentially, the principal's problem is to minimize the aggregate security subject to the constraints: $\forall k \in \{1,2,\ldots,n\}$,

$$Q_k \le \int_0^K E\left[S_k[z] \mid \hat{\theta}, M\right] f_k(M)\, dM.$$

It is not hard to see that the constraints must be binding for the optimal security bundle. An alternative way to think about the problem is that if we can relax the constraints while holding $\sum_{k=1}^{n} S_k[z]$ fixed, we can further shrink $\sum_{k=1}^{n} S_k[z]$. To shed light on the potential way to relax the constraints, we combine all the constraints into the following aggregate one

$$K \le \sum_{k=1}^{n} \int_0^{\infty} \{S_k[z] - S_k[0]\} \left\{\int_0^K g\left(z;\hat{\theta},M\right) f_k(M)\, dM\right\} dz.$$

The first term in the integral is a security's payoff sensitivity when the project value increases from 0 to z, and the second term is marginal Type-k agents' perceived distribution of the project value, which is governed by their perception of participation. Lower perception of participation implies that the distribution concentrates more (less) on low (high) project values.

As argued in Section 2, the principal should differentiate agents with respect to both payoff sensitivity and perception of participation and induce an "*assortative matching*" between them. However, the subtlety here is that an agent's payoff sensitivity is not uniform and vary with the project value. This implies that a more delicate "*assortative matching*" could be favorable for the principal: offer agents with low perception of participation securities with high payoff sensitivity at low project values and low payoff sensitivity at high project values. Later, we will see that a tranching structure is the best at implementing such delicate "*assortative matching*" .

5.2 Security formats

First, we provide a simple observation regarding two types with same perception of participation.

Lemma 3. *If a security bundle has $\Delta_{k-1,k} = 0$ in equilibrium, it is equivalent for the principal to offer Type-$(k-1)$ and Type-k agents the same security $(S_{k-1}+S_k, Q_{k-1}+Q_k)$ while keeping other securities unchanged. Conversely, it is equivalent for the principal to split the agents of any type k into two types and offer them $\left(S'_k, Q'_k\right)$ and $\left(S''_k, Q''_k\right)$ respectively, as long as $S'_k + S''_k = S_k$, $Q'_k + Q''_k = Q_k$, and*

$$\frac{\int_0^\infty S'_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k, M\right) f_k\left(M\right) dM\right] dz}{Q'_k} = \frac{\int_0^\infty S''_k[z]\left[\int_0^\infty g\left(z;\hat{x}_k, M\right) f_k\left(M\right) dM\right] dz}{Q''_k}.$$

If $\Delta_{k-1,k} = 0$, by Proposition 2 and Proposition 3, $f_{k-1}(M) = f_k(M)$. Therefore, Type-$(k-1)$ and Type-k marginal agents evaluate their securities in the exactly same way. Essentially, they act like the same type, so the principal can directly merge them into one type without changing the equilibrium. Conversely, for Type-k agents, the principal can split their aggregate security into two different ones and offer to them. Such split does not change the equilibrium as long as the new securities are valued the same by marginal Type-k agents.

Lemma 3 implies that to derive the optimal security bundle with the fewest types, it suffices to focus on the security bundles with all $\Delta_{k-1,k}$ being positive. Based on this, Proposition 6 characterizes the formats of the optimal securities: they must constitute a tranching structure.

Proposition 6. *The optimal security bundle with the fewest types must contain a tranching structure such that*

- *agents with identical perception of participation are offered an identical tranche and*
- *agents with lower perception of participation are offered more senior tranches.*

That is, if $\{(S_k, Q_k)\}_{k=1}^n$ is optimal, it must satisfy $\Delta_{k-1,k} > 0$ and $S_k = T_{F_k} - T_{F_{k-1}}$ where $0 = F_0 < F_1 < \ldots < F_n$.

Proposition 6 follows two ideas. The first idea is that the principal and the marginal agents value cash flows differently. To be specific, for the k-th type, suppose its cutoff is $\hat{\theta}$. Let

$$W_k^A(z) \equiv \int_0^\infty g\left(z;\hat{\theta}, M\right) f_k\left(M\right) dM,$$
$$W^P(z) \equiv \int_{\hat{\theta}}^\infty g\left(z;\theta, K\right) h(\theta) d\theta.$$

According to Equation (16), $W_k^A(z)$ is the marginal Type-k agents' posterior probability that the project value is z and each of them receives $s_k[z]$. Hence, if $s_k[z]$ increases by 1, their expected payoff increases by $W_k^A(z)$. $W^P(z)$ has a similar interpretation from the principal's perspective. According to Equation (20), if $S_k[z]$ increases by 1, the expected payment to Type-k agents increases by $W^P(z)$. Due to the the strict monotone likelihood ratio property (SMLRP) of the distribution of the project value $g(z;\theta,K)$ and the distribution of the noise $\phi(\cdot)$, $W_k^A(z)$ and $W^P(z)$ satisfy the following properties.

Lemma 4. *For any k and $z_1 < z_2$,*

$$\frac{W^P(z_2)}{W^P(z_1)} > \frac{W_k^A(z_2)}{W_k^A(z_1)} \text{ and } \frac{W_k^A(z_2)}{W_k^A(z_1)} > \frac{W_{k-1}^A(z_2)}{W_{k-1}^A(z_1)}.$$

The first implication of Lemma 4 is that the principal cares more about the security payment at high project values than marginal agents. From the principal's perspective, conditional on a positive mass of Type-k agents participating, the state is at least $\hat{\theta}$ almost surely, and the mass of participating agents is at least K almost surely. So, the project value is higher than that expected by the marginal Type-k agents. Another way to understand the difference is that the principal is concerned about the total payment to all participating agents including the marginal ones as well as those observing higher signals. The security payment of the latter ones is more at high project values.

The second implication of Lemma 4 is that marginal Type-k agents care more about the payment at high project values than marginal Type-$(k-1)$ agents. Since $\hat{x}_{k-1} = \hat{x}_k\ \Delta_{k-1,k} > 0$, the former ones have almost the same view on the fundamental as the latter ones but higher perception of participation. As a result, the former ones perceive high project values to be more likely than the latter ones do.

The second idea that Proposition 6 follows is that to convince agents to participate in a cost-effective way, the principal should allocate the cash flow at certain project values to the agents who value it most. On the one hand, since marginal agents value cash flows at low project values more, the principal would be better off by giving the agents cash flows at low project values in exchange for those at high project values. However, this improvement is constrained by monotonicity of securities and the budget constraint. Therefore, the optimal security bundle should look like a senior tranche in aggregate. On the other hand, regarding the allocation of cash flows between Type-k and Type-$(k-1)$ agents, the principal should allocate more cash flows at low (high) project values to Type-$(k-1)$ (Type-k) agents. Due to the same constraint mentioned above, this idea naturally implies that the optimal security bundle should constitute a tranching structure.

To see why a tranching structure is optimal more concretely, let's take a two-type security bundle $\{(S_k,Q_k)\}_{k=1}^2$ as an example. Suppose that $\Delta_{1,2} < +\infty$. In Panel (a) of Figure 1, the horizontal axis represents the project value z; the dotted line represents the 45-degree line; the two solid lines represent the two securities S_1 and S_2. Now, instead of S_1, Type-1 agents are offered S_1'.

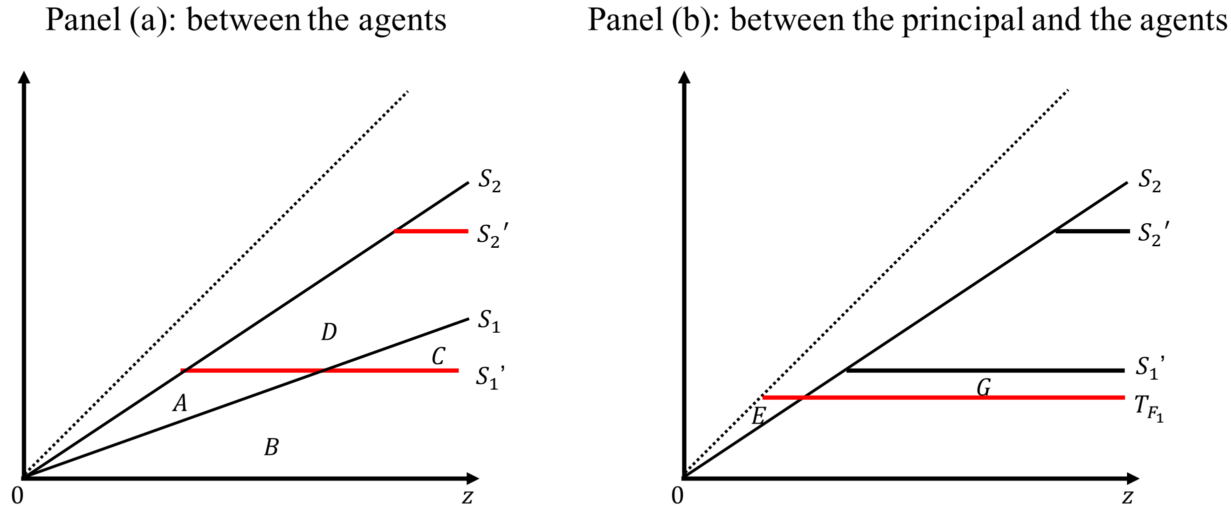

Figure 1: The Tranching Structure

To ensure that Type-1 agents still have the cutoff $\hat{\theta}$, Part A and Part C must be valued the same from the perspective of the marginal Type-1 agents. If we keep the aggregate security unchanged, Type-2 agents lose Part A but receive Part C. Since $W_2^A(z)/W_1^A(z)$ is strictly increasing in z, Part C implies a strictly higher expected payment than Part A from the perspective of the marginal Type-2 agents. That means, the firm can offer a security S_2', which specifies a strictly lower payment than $S_1 + S_2 - S_1'$ does, to Type-2 agents while still having them participate almost surely in the state $\hat{\theta}$. Hence, the total expected payment is reduced. Further, as shown in Panel (b) of Figure 1, the principal can offer a senior tranche T_{F_1} to Type-1 agents instead of S_1' such that Part E and Part G are valued the same from their perspective. Since $W^P(z)/W_1^A(z)$ is strictly increasing in z, Part E implies a strictly lower expected payment to Type-1 agents than Part G does from her perspective. As such, the total expected payment is reduced further. Iterating the same procedure, the principal can be better off by offering Type-2 agents a senior tranche of the residual project value $T_{F_2}[z - T_{F_1}[z]]$, which is also a junior tranche of the whole project value $(T_{F_2} - T_{F_1})[z]$.

Two comments about Proposition 6 are in order. First, the optimality of tranching structures does not rely on vanishing noise because the principal and marginal agents value cash flows differently in the same way as above for any σ. Second, Proposition 6 implies tranching structures but not necessarily ones with multiple tranches. It does not rule out the possibility that the optimal security bundle offers a senior tranche to all agents and they have the same perception of participation.

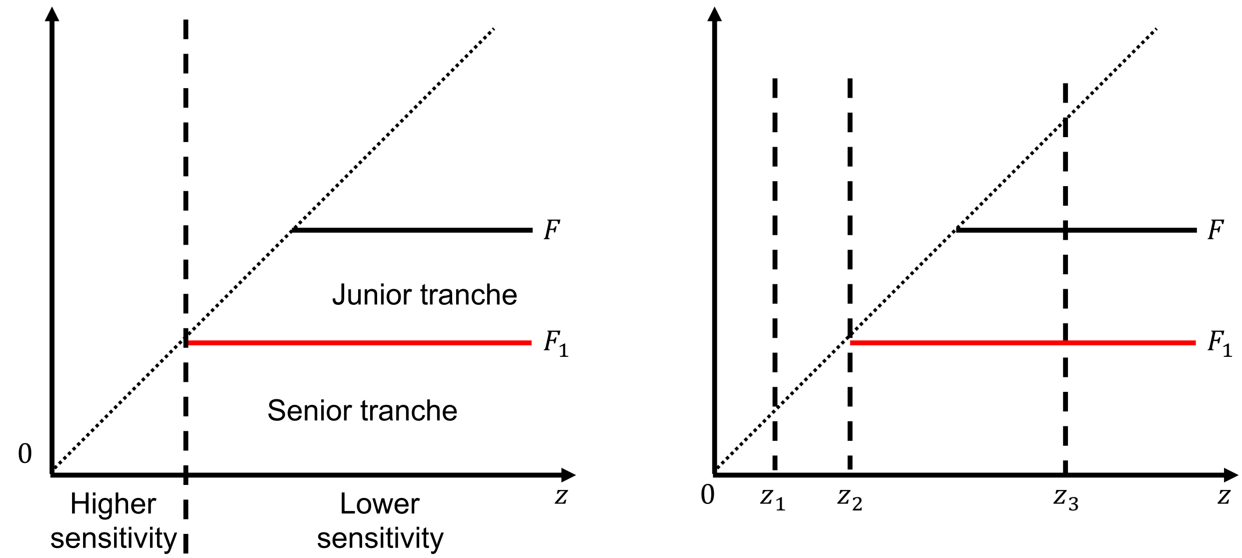

Figure 2: Positive Premium due to Miscoordination

5.3 The number of the types

In this subsection, we determine the number of the types of securities in the optimal security bundle with the fewest types. This feature is important because it directly indicates whether and to what extent differentiating agents is desirable. As discussed in Section 2 and Section 5.1, security design should implement an "*assortative matching*" strategy. A simple intuition is that a higher number of the types enables a finer way to differentiate agents and implement assortative matching. Proposition 7 confirms this intuition and indicates that the optimal security bundle should use up all available types.

Proposition 7. *The optimal security bundle with the fewest types of securities must contain N types.*

To illustrate the intuition of Proposition 7, we take $N = 2$ as an example. Suppose Proposition 7 does not hold and the optimal security bundle with the fewest types has only one type $\{(S,K)\}$ with an equilibrium cutoff $\hat{\theta}$. As implied by Proposition 6, it must be a senior tranche, so $S = T_F$. According to Lemma 3, we can split (T_F, K) into a senior tranche (T_{F_1}, Q_1) and a junior tranche $(T_F - T_{F_1}, K - Q_1)$ such that the mass K of agents behave in the same way as they do in the original equilibrium: $\hat{x}_1 = \hat{x}_2 = \hat{\theta}$ and $\Delta_{1,2} = 0$. Panel (a) of Figure 2 demonstrates this split.

Consider a marginal increase of $\Delta_{1,2}$ from 0. It pushes $f_1(M)$ to shift leftward and $f_2(M)$ to shift rightward, while keeping fixed their weighted sum

$$Q_1 f_1(M) + (K - Q_1) f_2(M).$$

As a result of the shift in perception of participation, Type-1 agents now demand a better security or $F_1' > F_1$ to participate in the state $\hat{\theta}$, and Type-2 agents now would be willing to accept $T_{F'} - T_{F_1'}$. A general insight in the literature of security design is that a senior tranche is less sensitive to the project value. If it is true, a small positive $\Delta_{1,2}$ per se will hurt Type-1 agents to a lesser extent than it benefits Type-2 agents, so $\left\{(T_{F_1'}, Q_1), (T_{F'} - T_{F_1'}, K - Q_1)\right\}$ strictly dominates $\{(T_F, K)\}$.

However, this insight does not hold for all ranges of project values. Specifically, in the 45-degree region of the senior tranche, the junior tranche is always worth 0 and has lower payoff sensitivities. Take as an example the three points in Panel (b) of Figure 2, z_1, z_2, and z_3 that satisfy $z_1 < z_2 = F_1 < F < z_3$. The senior tranche has lower payoff sensitivity than the junior tranche between z_2 and z_3 but higher one between z_1 and z_2.

Since the senior tranche has both a higher-sensitivity region and a lower-sensitivity region, the net impact of a marginal increase of $\Delta_{1,2}$ is ambiguous in general. When $f_1(M)$ shifts leftward, the distribution of the project value from marginal Type-1 agents' perspective also shift leftward. For example, we can imagine that some probability flows from z_2 and z_3 to z_1. Correspondingly, from marginal Type-2 agents‘ perspective, the same amount of probability flows from z_1 and z_2 to z_3. The senior tranche has lower (higher) payoff sensitivity between z_1 and z_3 (z_2 and z_3) than the junior tranche, so the flow of probability between them decreases (increases) F'. In general, it is hard to say which kind of flows dominate because it depends on the function of the project value g and the distribution of the noise ϕ.

However, we can always choose to carve out a sufficiently small senior tranche. With a smaller F_1, the senior tranche has a smaller 45-degree region. That means, the flow of probability between points like z_1 and z_3 is smaller while that between points like z_2 and z_3 is bigger. As such, there always exists a small senior tranche such that the flows that reduce F' dominate at margin so that $\partial F'/\partial \Delta_{1,2} < 0$ holds at $\Delta_{1,2} = 0$. Another way to understand this result is that a sufficiently small senior tranche is very close to a risk-free security that has zero payoff sensitivity everywhere, so the leftward shift in $f_1(M)$ does little harm to Type-1 agents but the rightward shift in $f_2(M)$ still does substantial good to Type-2 agents. Here, we can see the importance of the assumption $G(0;\theta,K) = 0$. When $G(0;\theta,K) > 0$, a senior tranche would always differ from a risk-free security substantially no matter how small it is.

5.4 The Bottom Line

To summarize, when multiple types of securities can be offered, the optimal security bundle uses multiple tranches to intentionally differentiate agents with respect to perception of participation and payoff sensitivity and match them positively. The motivation for such multi-tranche structures is to mitigate the adverse impact of miscoordination.[20]

The tranching structure is optimal because it polarizes the payoff sensitivity of different types so that they have very different comparative advantages to bear the adverse impact of miscoordination. A senior (junior) tranche has comparatively higher sensitivity at low (high) project values and should be offered to agents with low (high) perception of participation. Notably, the differentiation with respect to payoff sensitivity is closely associated with the differentiation with respect to perception of participation. When agents have the same perception of participation, as implied by Lemma 3, how their payoff sensitivity is differentiated does not matter: only their aggregate security matters. When agents have heterogeneous perception of participation, as implied by Proposition 6, tranches should be assigned to agents according to perception of participation.

Proposition 7 further confirms that it is desirable for the principal to differentiate agents in the finest way. It should be clear that this result relies heavily on that proper security formats can be used. In Section 6.1, we discuss a case with a restriction on security formats in practice. There differentiating agents is not favorable because the restriction prevents the beneficial "assortative matching". In a word, the interaction between the differentiation in the two dimensions is crucial to the optimal security bundle.

6 Discussion

6.1 Collinear securities

Here, we aim to make it clear that the desirability of differentiation relies on proper design of security formats. To illustrate this point, we consider the problem with a kind of restriction on security formats common in practice. In many contexts, securities can be differentiated but have to be collinear. For example, when a firm grants shares to important employees, the actual payoffs can be differentiated by heterogeneous numbers of shares but they are all proportional to the firm's performance. Similarly, in syndicate financing, an entrepreneur issues the same security to investors but may offer them different upfront fees. A natural question is, if only collinear securities can be offered, is differentiating agents desirable for the principal?

Proposition 8. *If the principal can only offer collinear securities, it is optimal for her to offer agents identical securities.*

Proposition 8 implies that it is actually strictly worse to differentiate agents if their securities have to be collinear. To see the intuition, let's take a look at the case with two-type bundles.

[20] It can be shown that when agents can coordinate perfectly, offering all agents an identical security can always be optimal in the baseline setup.

Consider the bundle $\{(Q_i s_i, Q_i)\}_{i=1,2}$ with $s_i[\cdot] = s[\cdot]/p_i$ that induce a common cutoff $\hat{\theta}$. Then the aggregate security is $S[\cdot] \equiv (Q_1/p_1 + Q_2/p_2)\, s[\cdot]$. If $p_1 < p_2$, Type-1 agents are offered a better security, so they must have lower *perception of participation.* However, they also have higher *payoff sensitivity.* For any $z > 0$,

$$s_i[z] - s_i[0] = \{s[z] - s[0]\}/p_i,$$

which is uniformly greater for a smaller p_k. Due to the restriction to collinear securities, differentiation can only lead to negative matching between perception of participation and payoff sensitivity, so it is not desirable.

In addition to manifesting that differentiation is not desirable on its own, this special case also helps contrast our model to the existing literature of contracting with externalities, especially Segal (2003), Winter (2004), and Halac et al. (2020). They capture strategic risk by focusing on the worst equilibrium of a coordination game with perfect information. Such combination of equilibrium selection and the information structure prohibits agents from providing assurance in a mutual way, so differentiation of agents' payoff is always preferred. Another recent paper, Halac et al. (2021), allows the principal to create rank uncertainty among agents, through which the principal can build mutual assurance among agents. They find that differentiation of agents' payoff is not preferred if agents are ex ante homogeneous. Different from theirs, our paper consider a coordination game with imperfect information and is the first to highlight whether differentiation of agents' payoffs is desirable depends on the payoff structures the principal can offer to agents.

6.2 Should differentiation be taken to extremes?

We have shown that the optimal security design induces differentiation in agents' perception of participation. A natural speculation is that it might be optimal to take such differentiation to extremes: let all $\Delta_{k-1,k}$ be infinity. It corresponds to the case where there is no strategic uncertainty between any two types of agents: marginal Type-$(k-1)$ agents think all Type-k agents reject almost surely and the marginal Type-k agents think all Type-$(k-1)$ agents participate almost surely. However, this speculation is not correct.

To quickly see this point, let's revisit the example in Section 2. According to the previous analysis, we know that for any $\Delta \geq 0$, it is the optimal to give Bank 1 the most senior part, so $c_1 = C$ and $c_2 = 0$. By Equation (7) and Equation (8), we obtain that the total face value is a function of Δ, i.e.,

$$
\begin{aligned}
D(\Delta) &\equiv d_1 + d_2 \\
&= C + \frac{1-C}{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(\Delta)} + \frac{1}{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(-\Delta)}.
\end{aligned}
$$

Taking the derivative with respect to Δ, we obtain

$$
\begin{aligned}
D'(\Delta) = &-\frac{df(\Delta)}{d\Delta} [P_1(\hat{x}) - P_0(\hat{x})] \times \\
&\left(\frac{1-C}{\{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(\Delta)\}^2} - \frac{1}{\{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(-\Delta)\}^2} \right)
\end{aligned}
$$

As $\Delta \to +\infty$, $f(\Delta) \to 0$ and $f(-\Delta) \to 1$. Hence, for $\Delta = +\infty$ to minimize $D(\Delta)$, we need

$$
\frac{1-C}{P_0(\hat{x})^2} - \frac{1}{P_1(\hat{x})^2} < 0.
$$

Apparently, this inequality does not hold when $P_0(\hat{x})$ is small relative to $P_1(\hat{x})$.

To see the intuition in a sharp way, let's assume $P_0(\hat{x}) \approx 0$ and $P_1(\hat{x}) \approx 1$. To induce a marginal increase in Δ, the firm needs to increase Bank 1's face value from d_1 to $d_1 + \delta_1$ because his perceives a lower success probability upon observing his cutoff. For Bank 1, this increase in the face value is worth almost

$$
\delta_1 \{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(\Delta)\}.
$$

Similarly, the firm can reduce Bank 2's face value from d_2 to $d_2 - \delta_2$, and for Bank 2, this decrease is worth almost

$$
\delta_2 \{P_0(\hat{x}) + [P_1(\hat{x}) - P_0(\hat{x})] \cdot f(-\Delta)\}.
$$

When $f(\Delta)$ is small, Bank 1 thinks that he is unlikely to get fully repaid, so one unit of increase in the face value of his debt benefits him negligibly. However, Bank 2 thinks that he is likely to get fully repaid, so one unit of decrease in the face value of his debt hurts him substantially. As a result of the difference in the perceived success probability, δ_1 is larger than δ_2. That means, a marginal increase in Δ increases the total face value.

Note that when $\Delta = 0$, the two banks perceive the same probability of getting fully repaid. One unit of change in the face value means the same for them at margin. Therefore, their difference in payoff sensitivity becomes the dominant factor.

To sum up, the tension is that increasing the face value of a senior tranche mainly increases its value at high project values, but senior tranche holders, who have low perception of participation, do not value cash flows at high project values much. Therefore, further reducing senior-tranche holders' perception of participation may require a large increase in the face value of the senior tranche.

6.3 General production technology

In the baseline setup, we deliberately assume that the production technology takes a form of generalized regime change to simplify the illustration. Here, we show that our main results still hold with more general production technology. We allow $g(z;\theta,K)$ to have multiple but a finite number of regime changes, i.e.,

$$g(z;\theta,K) = g_i(z;\theta,K), \text{if } \lambda_i(\theta) \le K < \lambda_{i+1}(\theta),$$

where $\lambda_i(\theta)$ is weakly decreasing in θ and strictly increasing in i. We still assume $g_i(z;\theta,K)$ is Lipschitz continuous but impose no restriction on the Lipschitz constants. Specifically, $\exists R > 0$, s.t. for any i,

$$|E_i[z|\theta_1,K_1] - E_i[z|\theta_2,K_2]| \le R|\theta_1 - \theta_2| + R|K_1 - K_2|.$$

That means, we abandon the assumption that agents' participation does not generate positive surplus unless it induces regime change.

In this setup, Proposition 2, Proposition 3, and Proposition 4 still hold, but Proposition 5 does not. The optimal security bundle may induce agents to have different cutoffs in the limit case. Suppose that the set of all cutoffs have T elements, say $\theta_1 < \theta_2 < \ldots < \theta_T$. In addition, for notional convenience, we let $-\theta_0 = \theta_{T+1} = +\infty$. The mass of participating agents is essentially a step function of the state θ

$$K\left(\theta;\{K_t,\theta_t\}_{t=1}^T\right) \equiv \sum_{t=1}^{T} K_t \cdot 1\{\theta_t < \theta \le \theta_{t+1}\}, \tag{PS}$$

where K_t is equal to the mass of agents whose cutoff is no more than θ_t, i.e., $K_t = \sum_{\{k|\hat{x}_k \le \theta_t\}} Q_k$. $K\left(\theta;\{K_t,\theta_t\}_{t=1}^T\right)$ is referred to as a T-cutoff participation scheme throughout the paper. Note that T is weakly smaller than the number of types because agents of different types can have the same cutoff in the limit case.

The principal's problem can be divided into two steps. The first step is to pick a participation scheme. Since the expected project value is completely determined by the participation scheme, the second step is to minimize the total expected payment to agents, given that the participation scheme is implemented. We can show that given any participation scheme, the optimal security bundle implementing it must satisfy the properties specified in Proposition 6 and Proposition 7.

6.4 Offering menus to agents

The baseline setup assumes that the principal can offer an agent one security. Here, we show that this is without loss of generality because the principal does not benefit from offering more than one security to any agent.

Suppose that the principal offers n types of menus to agents. Denote the maximum payoff that a Type-k agent can receive at the project value z by choosing a security from his menu by $s_k^{\max}[\cdot]$. Due to the budget constraint, we have

$$\sum_{k=1}^{n} Q_k s_k^{\max}[z] \leq z.$$

It is not hard to see that each agent still follows a cutoff strategy to participate. Denote their cutoffs by $\{\hat{x}_k\}_{k=1}^{n}$. Suppose that upon observing $\hat{x}_1$, Type-1 agents participate and choose the security $s_1^1[\cdot]$.

We claim that the principal is at least weakly better off by offering the single security $s_1^1[\cdot]$ to Type-1 agents while holding the menus offered to other agents fixed. This change does not violate the budget constraint and thus is feasible. Notice that $s_1^1[\cdot]$ must satisfy

$$\int_0^{\infty} s_1^1[z] \left[\int_0^{\infty} g(z;\hat{x}_1, M) f_1(M)\, dM \right] dz = 1.$$

If the principal offers $s_1^1[\cdot]$ instead, all agents' participation still follow the same cutoff strategies. Therefore, offering Type-1 agents more securities other than $s_1^1[\cdot]$ does not change the expected project value, but gives them more options to maximize the expected security payment that the principal pays to them. Therefore, the principal does not benefit from offering more options.

6.5 Zero contracting premium due to miscoordination

In the baseline model, we assume that the lowest possible project value is always zero irrespective of the state of the economy θ and the level of participation K. As a result, the contracting premium due to miscoordination, which is captured by the difference between the expected security payments to agents and their participation costs, is always positive, and thus the principal always prefer a finer differentiation strictly. In practice, it is also possible that as θ or K gets higher, the lowest possible project value increases accordingly. This opens up the possibility that the principal can use a security bundle to achieve zero contracting premium. Since such a bundle has achieved the theoretically lowest cost, the principal does not benefit from a finer differentiation. In this subsection, we derive the condition for such a security bundle to exist and characterize the minimum number of types required by it.

For any θ and K, define

$$V(\theta,K) \equiv \inf\{z|G(z;\theta,K) > 0\}.$$

$V(\theta,K)$ represents the effectively lowest possible project value. For any participation scheme $K\left(\theta;\{(K_t,\theta_t)\}_{t=1}^{T}\right)$, define

$$\tilde{V}_t(x) \equiv V(\theta_t, x+K_{t-1}) - K_{t-1}$$

and

$$\tilde{V}_t^{(n)}(x) \equiv \underbrace{\tilde{V}_t \circ \tilde{V}_t \circ \cdots \tilde{V}_t}_{n}(x).$$

Proposition 9. *The participation scheme $K\left(\theta;\{(K_t,\theta_t)\}_{t=1}^{T}\right)$ can be implemented by a finite-type security bundle with zero premium, if and only if $\tilde{V}_t(x) > x$ for any $t \in \{1,2,\dots,T\}$ and $x \in [0, K_t - K_{t-1})$. The minimum number of types to achieve zero premium is $n^* = \sum_{t=1}^{T} n_t^*$ where $n_t^* \equiv \min\left\{n|\tilde{V}_t^{(n)}(0) \geq K_t - K_{t-1}\right\}$.*

To illustrate the intuition of Proposition 9, we take the types with the lowest cutoff θ_1 as an example and see how n_1^* is determined. Without loss of generality, we assume that this security bundle contains a tranching structure as in Proposition 6. Suppose the first l types have the cutoff θ_1 and they all have zero premium. Zero premium requires that $F_k = \sum_{i=1}^{k} Q_i$. That means, any marginal Type-k agents can receive $F_k - F_{k-1} = Q_k$ in almost all scenarios with positive probability. From the perspective of the marginal Type-k agents, at most the first $(k-1)$ types of agents participate almost surely, so the worst scenario with positive probability is no better than only mass $\sum_{j=1}^{k-1} Q_j$ of agents participating. Therefore,

$$V\left(\theta_1, \sum_{j=1}^{k-1} Q_j\right) - F_{k-1} \geq T_{F_k}\left[V\left(\theta_1, \sum_{j=1}^{k-1} Q_j\right)\right] - F_{k-1} \geq Q_k$$
$$\Rightarrow \tilde{V}_1\left(\sum_{j=1}^{k-1} Q_j\right) > \sum_{j=1}^{k} Q_j.$$

Iterating this inequality, we obtain that $\sum_{j=1}^{k} Q_j \leq \tilde{V}_1^{(k)}(0)$. Therefore, $\tilde{V}_1^{(k)}(0)$ is the upper bound of the mass of the first k types of agents if zero premium is required. n_1^* is the minimum number such that this upper bound exceeds the total mass of the agents with the cutoff θ_1.

Figure 3 illustrates one security bundle that achieves zero premium with $n_1^* = 3$ types. The first type features mass $\tilde{V}_1(0)$ of agents sharing the senior tranche $T_{V(\theta_1,0)}$. Upon observing θ_1, each of them can receive

$$\frac{V(\theta_1,0)}{\tilde{V}_1(0)} = 1$$

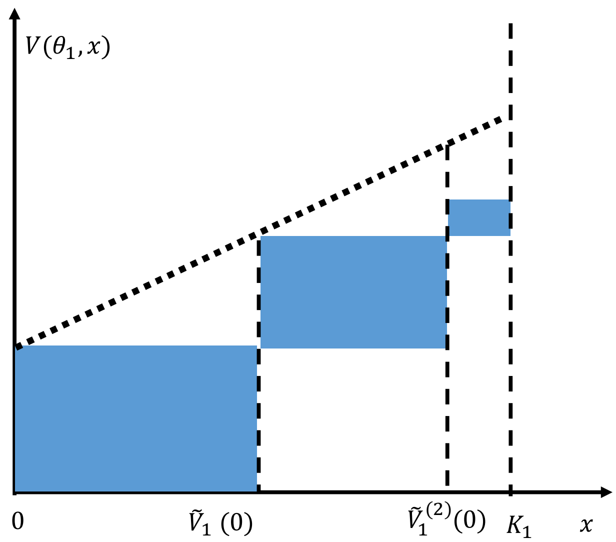

Figure 3: The security bundle achieving zero Premium

almost surely, so they choose to participate irrespective of others' decisions. The second type features mass $\tilde{V}_1^{(2)}(0) - \tilde{V}_1(0)$ of agents sharing the junior tranche $T_{V(\theta_1, \tilde{V}_1(0))} - T_{V(\theta_1, 0)}$. Upon observing θ_1, they know Type-1 agents participate almost surely, so each of them can receive

$$\frac{V(\theta_1, \tilde{V}_1(0)) - V(\theta_1, 0)}{\tilde{V}_1^{(2)}(0) - \tilde{V}_1(0)} = 1$$

almost surely. Based on the speculation about Type-1 agents, they choose to participate. The n_1^*-th type features mass $K_1 - \tilde{V}_1^{(2)}(0)$ of agents sharing the junior tranche $T_{K_1} - T_{V(\theta_1, \tilde{V}_1(0))}$. Conditional on the participation of the first two types of agents, they choose to participate as well.

6.6 Applications

We have formulated our model as a principal motivating agents to participate in a project. There are various examples that may fit this description. We next discuss some applications.

Corporate finance. In corporate finance, investors' payoffs are usually linked to the state of the firm, which is affected by other investors' contribution. Based on the strategic risk between investors, a number of papers study debt rollover risk. Our results directly imply that to mitigate the adverse impact of strategic risk, the firm should differentiate investors by offering different securities. Some investors have payoffs less sensitive to the state of the firm, so they are not so concerned about strategic risk and become eager to invest. Their eagerness further reassures other investors so that other investors would also like to invest despite high sensitivities of their payoffs to the state of the firm. Such differentiation can be implemented by a debt-seniority structure or uneven allocation of collateral among investors.

Financial stability. Strategic risk also undermines the stability of financial systems. A large literature sheds light on how regulators use policy tools to bolster financial stability. A companion paper, Dai et al. (2023), studies how to design bank disclosures to achieve this goal. The paper finds that revealing banks' heterogeneity may effectively alter banks' investors' *perception of participation* and *payoff sensitivity to participation*. Following similar intuition, the paper concludes that revealing banks' heterogeneous vulnerabilities to systemic risk to some extent can make the whole banking system more robust but revealing banks' heterogeneous idiosyncratic shortfall of funds does not. Our model also provides guidance for the use of capital requirement on banks. Capital requirement increases banks' resistance to adverse shock but also restricts banks' ability to provide welfare-improving services. Given a sufficiently robust banking system, regulators would prefer to economize on the use of capital requirement. Applying our results to this setting suggests that heterogeneous capital requirement should be imposed on banks. The investors of the banks with stricter capital requirement are less concerned about systemic risk, and their confidence in their banks further bolsters other investors' confidence in the whole banking system.

Employee compensation. A firm's performance depends on the effort of all employees. The design of employee compensation should take into consideration employees' concern about others' effort. Different from the existing literature (Segal, 2003; Winter, 2004; Halac et al., 2020), our results imply that a simple differentiation of employees in terms of high or low rewards is not helpful; employee compensation should be differentiated with respect to its sensitivity to the firm's performance. Roughly, employee compensation consists of three main parts: salary, bonus, and stock & option. Salary is the least sensitive to the firm's performance, stock & option is the most sensitive, and bonus is in the middle. The differentiation with respect to sensitivity can be implemented by offering different combination of these parts to employees.

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A Additional Lemmas

Lemma 5.

$|E[S_k[z]|\theta_1,K_1]-E[S_k[z]|\theta_2,K_2]| \leq R_k|E[z|\theta_1,K_1]-E[z|\theta_1,K_2]|+R_k|E[z|\theta_1,K_2]-E[z|\theta_2,K_2]|.$

Proof. Consider $\theta_1>\theta_2$. $g(z;\theta_1,K)/g(z;\theta_2,K)$ is strictly increasing in z. Let $\tilde{z}\equiv \inf\{g(z;\theta_1,K)/g(z;\theta_2,K)>1$

$$
\begin{aligned}
&|E[S_k[z]|\theta_1,K]-E[S_k[z]|\theta_2,K]| \\
=&\left|\int_{z>\tilde{z}} S_k[z]\left[g(z;\theta_1,K)-g(z;\theta_2,K)\right]dz-\int_{z<\tilde{z}} S_k[z]\left[g(z;\theta_2,K)-g(z;\theta_1,K)\right]dz\right|
\end{aligned}
$$

Note that

$$\int_{z>\tilde{z}}\left[g(z;\theta_1,K)-g(z;\theta_2,K)\right]dz=\int_{z<\tilde{z}}\left[g(z;\theta_2,K)-g(z;\theta_1,K)\right]dz.$$

So,

$$
\begin{aligned}
&|E[S_k[z]|\theta_1,K]-E[S_k[z]|\theta_2,K]| \\
=&\frac{\int_{z_1>\tilde{z}}\int_{z_2<\tilde{z}}(S_k[z_1]-S_k[z_2])\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]\left[g(z_1;\theta_1,K)-g(z_1;\theta_2,K)\right]dz_2dz_1}{\int_{z_2<\tilde{z}}\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]dz_2} \\
\leq&\frac{\int_{z_1>\tilde{z}}\int_{z_2<\tilde{z}}R_k(z_1-z_2)\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]\left[g(z_1;\theta_1,K)-g(z_1;\theta_2,K)\right]dz_2dz_1}{\int_{z_2<\tilde{z}}\left[g(z_2;\theta_2,K)-g(z_2;\theta_1,K)\right]dz_2} \\
=&|E[R_kz|\theta_1,K]-E[R_kz|\theta_2,K]| \\
\leq&R_k|E[z|\theta_1,K]-E[z|\theta_2,K]|.
\end{aligned}
$$

Likewise, for $K_1>K_2$,

$$|E[S_k[z]|\theta,K_1]-E[S_k[z]|\theta,K_2]|\leq R_k|E[z|\theta,K_1]-E[z|\theta,K_2]|.$$

Then we obtain

$$
\begin{aligned}
&|E[S_k[z]|\theta_1,K_1]-E[S_k[z]|\theta_2,K_2]| \\
=&|E[S_k[z]|\theta_1,K_1]-E[S_k[z]|\theta_1,K_2]+E[S_k[z]|\theta_1,K_2]-E[S_k[z]|\theta_2,K_2]| \\
\leq&|E[S_k[z]|\theta_1,K_1]-E[S_k[z]|\theta_1,K_2]|+|E[S_k[z]|\theta_1,K_2]-E[S_k[z]|\theta_2,K_2]| \\
\leq&R_k|E[z|\theta_1,K_1]-E[z|\theta_1,K_2]|+R_k|E[z|\theta_1,K_2]-E[z|\theta_2,K_2]|.
\end{aligned}
$$

□

Lemma 6. *$\phi(\cdot)$ is bounded, and $\lim_{|\varepsilon|\to+\infty}\phi(\varepsilon)=0$.*

Proof. Due to SMLRP,

$$\begin{aligned}\frac{\phi(\varepsilon+a)}{\phi(\varepsilon+2a)} &> \frac{\phi(\varepsilon)}{\phi(\varepsilon+a)}\\ \Leftrightarrow -\log\phi(\varepsilon+a) &< \frac{-\log\phi(\varepsilon)-\log\phi(\varepsilon+2a)}{2}\end{aligned}$$

which means $-\log\phi(\cdot)$ is strictly mid-point convex. Since $\phi(\cdot)$ is continuous, so is $-\log\phi(\cdot)$. According to Jensen (1906), $-\log\phi(\cdot)$ is strictly convex.

Since $\int_{-\infty}^{+\infty}\phi(\varepsilon)d\varepsilon=1$, $\phi(\cdot)$ cannot be always increasing or decreasing, so $\phi(\cdot)$ must be first increasing and then decreasing. Therefore, $\phi(\cdot)$ is bounded, and $\lim_{|\varepsilon|\to+\infty}\phi(\varepsilon)\to 0$. □

B Proofs

Proof of Proposition 1

By Equation (7) and Equation (8), we obtain that the total face value is a function of (α,Δ), i.e.,

$$D(\alpha,\Delta)\equiv d_1+d_2=2c+\frac{1-(c+\alpha)}{p_1(\Delta)}+\frac{1-(c-\alpha)}{p_2(\Delta)}.$$

On the one hand, since $p_1(0)=p_2(0)$, $D(\alpha,0)$ does not change with α for $\alpha<\min\{c,1-c\}$. On the other hand, the derivative of $D(\alpha,\Delta)$ with respect to Δ at $\Delta=0$ is

$$\left.\frac{\partial D(\alpha,\Delta)}{\partial\Delta}\right|_{\Delta=0}=\frac{P_1(\hat{x})-P_0(\hat{x})}{p_1(0)^2}\cdot\left(\alpha\cdot\left.\frac{df(\Delta)}{d\Delta}\right|_{\Delta=0}\right)+O(\sigma).$$

Therefore, for sufficiently small σ, there always exists a positive pair (c,Δ) such that

$$D(\alpha,\Delta)<D(\alpha,0)=D(0,0).$$

Proof of Lemma 2

For Type-k agents, the marginal payoff of participating is $E\left[s_k[z]|\theta,K\right]-1$. On the one hand,

$$E\left[s_k[z]|\theta,K\right]-1=E\left[S_k[z]-S_k[0]|\theta,K\right]/Q_k-1\leq E\left[z|\theta,1\right]R_k/Q_k-1.$$

The right-hand side converges to -1 as θ goes to $-\infty$, so there exists $\underline{\theta}$ such that $E[s_k[z]|\theta, K] - 1 < 0$ for $\theta \leq \underline{\theta}$ irrespective of K. On the other hand, for $\hat{z}$ satisfying $s_k[\hat{z}] > 1$,

$$E[s_k[z]|\theta, K] - 1 \geq s_k[\hat{z}][1 - G(\hat{z}; \theta, 0)] - 1.$$

The right-hand side converges to $s_k[\hat{z}] - 1$ as θ goes to $+\infty$, so there exists $\overline{\theta}$ such that $E[s_k[z]|\theta, K] - 1 > 0$ for $\theta \geq \overline{\theta}$ irrespective of K.

Proof of Proposition 2

According to the breakeven condition of the marginal Type-k agents, Equation (14), we obtain

$$\int_{-\infty}^{\infty} \left[\int_0^{\infty} (S_k[z] - Q_k)\, g(z; \theta, M^{\sigma}(\theta))\, dz \right] \frac{1}{\sigma} \phi\left(\frac{\hat{x}_k^{\sigma} - \theta}{\sigma}\right) h(\theta) d\theta = 0.$$

Let $m_k = 1 - \Phi\left(\frac{\hat{x}_k^{\sigma} - \theta}{\sigma}\right)$ and $\Delta_{k,j}^{\sigma} \equiv \left(\hat{x}_j^{\sigma} - \hat{x}_k^{\sigma}\right)/\sigma$. Then

$$M^{\sigma}(\theta) = \sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j \Phi\left(\Phi^{-1}(1 - m_k) + \Delta_{k,j}^{\sigma}\right).$$

The above breakeven condition can rewritten as

$$\int_0^1 \left[\int_0^{\infty} (S_k[z] - Q_k) \Gamma(z; \theta, \{\Delta_{k,j}^{\sigma}\}_{j=1}^{n}, m_k) dz \right] dm_k = 0,$$

where $\theta = \hat{x}_k^{\sigma} - \sigma \Phi^{-1}(1 - m_k)$ and

$$\Gamma(z; \theta, \{\Delta_{k,j}^{\sigma}\}_{j=1}^{n}, m_k) \equiv g\left(z; \theta, \sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j \Phi\left(\Phi^{-1}(1 - m_k) + \Delta_{k,j}^{\sigma}\right)\right) h(\theta).$$

Part I: There always exists an infinite subsequence $\{\sigma_m\}_{m=1}^{+\infty}$ converging to 0 such that any $\{\hat{x}_k^{\sigma_m}\}_{m=1}^{+\infty}$ converges to $\hat{x}_k^0 \in (-\infty, +\infty)$ and any $\left\{\Delta_{j,k}^{\sigma_m}\right\}_{m=1}^{+\infty}$ converges to $\Delta_{j,k}^0 \in [-\infty, +\infty]$. Moreover, $\{(\hat{x}_k^0, \Delta_{j,k}^0)\}_{j,k \in \{1,2,\ldots,n\}}$ satisfy the equation system in Proposition 2.

The existence of such converging sequences is obvious. By Lemma 2, for sufficiently small σ_m, $\hat{x}_k^{\sigma_m} \in (\underline{\theta}, \overline{\theta})$, so $\hat{x}_k^0$ must be finite. In addition, if $\hat{x}_k^0 > (<) \hat{x}_{k-1}^0$,

$$\Delta_{k-1,k}^0 = \lim_{m \to \infty} \Delta_{k-1,k}^{\sigma_m} = \lim_{m \to \infty} \frac{\hat{x}_k^{\sigma_m} - \hat{x}_{k-1}^{\sigma_m}}{\sigma_m} = \lim_{m \to \infty} \frac{\hat{x}_k^0 - \hat{x}_{k-1}^0}{\sigma_m} = +\infty(-\infty),$$

and

$$\Delta^0_{j,k} = \lim_{m\to\infty} \Delta^{\sigma_m}_{j,k} = \lim_{m\to\infty} \sum_{i=j+1}^{k} \Delta^{\sigma_m}_{i-1,i} = \sum_{i=j+1}^{k} \Delta^0_{i-1,i}.$$

To confirm the existence of the solution to the equation system in Proposition 2, we will prove

$$\int_0^1 \left[\int_0^\infty (S_k[z] - Q_k)\,\Gamma(z; \hat{x}^0_k, \{\Delta^0_{k,j}\}^n_{j=1}, m_k) dz \right] dm_k = 0.$$

Step 1 We claim that for any σ and $\varepsilon > 0$, there exists $t_1 > 0$ such that

$$\left| \int_{\left|\Phi^{-1}(1-m_k)\right| \geq t_1} \left[\int_0^\infty (S_k[z] - Q_k)\,\Gamma(z; \theta, \{\Delta^\sigma_{k,j}\}^n_{j=1}, m_k) dz \right] dm_k \right| < \varepsilon$$

and

$$\left| \int_{\left|\Phi^{-1}(1-m_k)\right| \geq t_1} \left[\int_0^\infty (S_k[z] - Q_k)\,\Gamma(z; \hat{x}^0_k, \{\Delta^\sigma_{k,j}\}^n_{j=1}, m_k) dz \right] dm_k \right| < \varepsilon.$$

Note that

$$\begin{aligned}
&\left| \int_{\left|\Phi^{-1}(1-m_k)\right| \geq t_1} \left[\int_0^\infty (S_k[z] - Q_k)\,\Gamma(z; \theta, \{\Delta^\sigma_{k,j}\}^n_{j=1}, m_k) dz \right] dm_k \right| \\
\leq &\int_{\left|\frac{\hat{x}^\sigma_k - \theta}{\sigma}\right| \geq t_1} \left[\int_0^\infty (S_k[z] + Q_k)\, g\,(z; \theta, 1)\, h(\theta) dz \right] \frac{1}{\sigma} \sup\{\phi(\cdot)\} d\theta \\
\leq &\frac{1}{\sigma} \sup\{\phi(\cdot)\} \int_{\left|\frac{\hat{x}^\sigma_k - \theta}{\sigma}\right| \geq t_1} (E\,[z|\theta, 1] + Q_k)\, h(\theta) d\theta
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{\left|\Phi^{-1}(1-m_k)\right| \geq t_1} \left[\int_0^\infty (S_k[z] - Q_k)\,\Gamma(z; \hat{x}^0_k, \{\Delta^\sigma_{k,j}\}^n_{j=1}, m_k) dz \right] dm_k \right| \\
\leq &\int_{\left|\frac{\hat{x}^\sigma_k - \theta}{\sigma}\right| \geq t_1} \left[\int_0^\infty (S_k[z] + Q_k)\, g\left(z; \hat{x}^0_k, 1\right) h(\hat{x}^0_k) dz \right] \frac{1}{\sigma} \phi\left(\frac{\hat{x}^\sigma_k - \theta}{\sigma} \right) d\theta \\
\leq &\left(E\left[z|\hat{x}^0_k, 1\right] + Q_k\right) h(\hat{x}^0_k) \int_{\left|\frac{\hat{x}^\sigma_k - \theta}{\sigma}\right| \geq t_1} \frac{1}{\sigma} \phi\left(\frac{\hat{x}^\sigma_k - \theta}{\sigma} \right) d\theta
\end{aligned}$$

Since $\int_{-\infty}^{\infty} (E\,[z; \theta, 1] + Q_k)\, h(\theta) d\theta$ and $\int_{-\infty}^{\infty} \frac{1}{\sigma} \phi\left(\frac{\hat{x}^\sigma_k - \theta}{\sigma} \right) d\theta$ are both finite, such t_1 exists.

Step 2 We claim that there exists $\overline{\sigma}$ such that for any $\sigma_m < \overline{\sigma}$,

$$\left| \int_{\left|\Phi^{-1}(1-m_k)\right|<t_1} \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0, \{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n, m_k) - \Gamma(z;\theta, \{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n, m_k) \right] dz dm_k \right| < \varepsilon.$$

Since $\left|\Phi^{-1}(1-m_k)\right| < t_1$, $\left|\theta - \hat{x}_k^0\right| < \left|\hat{x}_k^0 - \hat{x}_k^{\sigma_m}\right| + \sigma t_1 \equiv t_2$. As $\sigma \to 0$, $t_2 \to 0$. Denote $\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}^{\sigma_m}\right)$ by M.

$$\begin{aligned}
&\left| \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0, \{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n, m_k) - \Gamma(z;\theta, \{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n, m_k) \right] dz \right| \\
\leq &\left| \int_0^\infty S_k[z] \left[g\left(z;\hat{x}_k^0, M\right) - g\left(z;\theta, M\right) \right] dz \right| h(\theta) + \int_0^\infty S_k[z] g\left(z;\hat{x}_k^0, M\right) dz \cdot \left|h(\hat{x}_k^0) - h(\theta)\right| + Q_k \left|h(\hat{x}_k^0) - h(\theta)\right| \\
\leq &\left| \int_0^\infty S_k[z] \left[g\left(z;\hat{x}_k^0, M\right) - g\left(z;\theta, M\right) \right] dz \right| h(\theta) + E\left[z|\hat{x}_k^0, 1\right] \cdot \left|h(\hat{x}_k^0) - h(\theta)\right| + Q_k \left|h(\hat{x}_k^0) - h(\theta)\right|
\end{aligned}$$

For $M \geq \lambda(\hat{x}_k^0 - t_2)$, since $M \geq \lambda(\hat{x}_k^0)$ and $M \geq \lambda(\theta)$,

$$\begin{aligned}
\left| \int_0^\infty S_k[z] \left[g\left(z;\hat{x}_k^0, M\right) - g\left(z;\theta, M\right) \right] dz \right| h(\theta) &= \left| \int_0^\infty S_k[z] \left[g_1\left(z;\hat{x}_k^0, M\right) - g_1\left(z;\theta, M\right) \right] dz \right| h(\theta) \\
&\leq R_k R t_2 \cdot \sup\{h(\cdot)\}.
\end{aligned}$$

Similarly, for $M < \lambda(\hat{x}_k^0 + t_2)$,

$$\begin{aligned}
\left| \int_0^\infty S_k[z] \left[g\left(z;\hat{x}_k^0, M\right) - g\left(z;\theta, M\right) \right] dz \right| h(\theta) &= \left| \int_0^\infty S_k[z] \left[g_0\left(z;\hat{x}_k^0, M\right) - g_0\left(z;\theta, M\right) \right] dz \right| h(\theta) \\
&\leq R_k R t_2 \cdot \sup\{h(\cdot)\}.
\end{aligned}$$

For $\lambda(\hat{x}_k^0 + t_2) \leq M < \lambda(\hat{x}_k^0 - t_2)$

$$\begin{aligned}
&\left| \int_0^\infty S_k[z] \left[g\left(z;\hat{x}_k^0, M\right) - g\left(z;\theta, M\right) \right] dz \right| h(\theta) \\
\leq &\left| \int_0^\infty S_k[z] \left[g_1\left(z;\hat{x}_k^0, M\right) - g_0\left(z;\hat{x}_k^0, M\right) \right] dz \right| h(\theta) + R_k R t_2 \cdot h(\theta) \\
\leq &R_k \left\{ E_1\left[z|\hat{x}_k^0, M\right] - E_0\left[z|\hat{x}_k^0, M\right] \right\} \sup\{h(\cdot)\} + R_k R t_2 \cdot \sup\{h(\cdot)\}
\end{aligned}$$

Note that $h(\theta)$ is continuous in θ. As $t_2 \to 0$, $\left|h(\hat{x}_k^0) - h(\theta)\right| \to 0$, $R_k R t_2 \cdot \sup\{h(\cdot)\} \to 0$, and the measure of m_k satisfying $\lambda(\hat{x}_k^0 + t_2) \leq M < \lambda(\hat{x}_k^0 - t_2)$ goes to 0, so our claim holds.

Step 3 We claim that there exists $\overline{\sigma}$ such that for any $\sigma_m < \overline{\sigma}$,

$$\left| \int_{\left|\Phi^{-1}(1-m_k)\right|<t_1} \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) - \Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k) \right] dzdm_k \right| < \varepsilon.$$

Here we prove that as $\sigma_m \to 0$,

$$\left| \Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^{\sigma_m}\right) - \Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right) \right|$$

uniformly converge to 0 for any m_k satisfying $\left|\Phi^{-1}(1-m_k)\right| < t_1$. If $\Delta_{k,j}^0$ is finite, it follows that $\Delta_{k,j}^{\sigma_m} \to \Delta_{k,j}^0$ and $\phi(\cdot)$ is bounded. If $\Delta_{k,j}^0 = +\infty$, $\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right) = 1$. Since $\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^{\sigma}\right)$ increases to 1 as $\Delta_{k,j}^{\sigma_m} \to +\infty$, for any $\delta > 0$, when σ_m is sufficiently small, $\Delta_{j,k}^{\sigma_m}$ can be large enough such that for any m_k satisfying $\left|\Phi^{-1}(1-m_k)\right| < t_1$,

$$\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^{\sigma_m}\right) \geq \Phi(-t_1+\Delta_{k,j}^{\sigma_m}) > 1-\delta.$$

The case of $\Delta_{k,j}^0 = -\infty$ follows a similar proof.

Step 4 For $\sigma_m < \overline{\sigma}$,

$$\begin{aligned}
&\left| \int_0^1 \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) - \Gamma(z;\theta,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) \right] dzdm_k \right| \\
\leq &\left| \int_{\left|\Phi^{-1}(1-m_k)\right|<t_1} \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) - \Gamma(z;\theta,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) \right] dzdm_k \right| + 2\varepsilon \\
\leq &3\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left| \int_0^1 \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) - \Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k) \right] dzdm_k \right| \\
\leq &\left| \int_{\left|\Phi^{-1}(1-m_k)\right|<t_1} \int_0^\infty (S_k[z]-Q_k) \left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k) - \Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k) \right] dzdm_k \right| + 2\varepsilon \\
< &3\varepsilon.
\end{aligned}$$

All combined, when m is sufficiently large,

$$
\begin{aligned}
&\left|\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k)dz\right] dm_k\right| \\
=&\left|\int_0^1 \int_0^\infty (S_k[z]-Q_k)\left[\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k)-\Gamma(z;\theta,\{\Delta_{k,j}^{\sigma_m}\}_{j=1}^n,m_k)\right] dzdm_k\right| \\
<&6\varepsilon.
\end{aligned}
$$

Since ε can be arbitrarily small,

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}_k^0,\{\Delta_{k,j}^0\}_{j=1}^n,m_k)dz\right] dm_k = 0.$$

Let

$$M=\sum_{j=1}^n Q_j-\sum_{j=1}^n Q_j\Phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right),$$

which ranges from 0 to $\sum_{j=1}^n Q_j$.

$$\frac{dm_k}{dM}=\frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q_j\phi\left(\Phi^{-1}(1-m_k)+\Delta_{k,j}^0\right)}=f\left(M;\{Q_j,\Delta_{k,j}^0\}_{j=1}^n\right).$$

So,

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,g\left(z;\hat{x}_k^0,M\right)dz\right] f\left(M;\{Q_j,\Delta_{k,j}^0\}_{j=1}^n\right)dM=0.$$

We have confirmed the existence of the solution to the equation system.

Part II: There is a unique solution $\{\hat{x}_k\}_{k=1}^n$ to the equation system.

Suppose $\{\hat{x}_k\}_{k=1}^n$ and $\{\hat{x}'_k\}_{k=1}^n$ both satisfy the equation system and they are different in at least one element. They have $\{\Delta_{k,j}\}_{j,k\in\{1,\ldots,n\}}$ and $\{\Delta'_{k,j}\}_{j,k\in\{1,\ldots,n\}}$ respectively.

Suppose there are types with $\hat{x}'_k>\hat{x}_k$ and they constitute the set $\mathscr{T}=\{\tau_1,\tau_2,\ldots,\tau_L\}$ where $\tau_1<\tau_2\ldots<\tau_L$. Consider $k\in\mathscr{T}$. Since

$$\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}'_k,\{\Delta'_{k,j}\}_{j=1}^n,m_k)dz\right] dm_k$$

is strictly decreasing in $\Delta'_{k,j}$, if $\Delta'_{k,j} \leq \Delta_{k,j}$ for any j, then

$$\begin{aligned}
&\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}'_k,\{\Delta'_{k,j}\}_{j=1}^n,m_k)dz\right] dm_k \\
> &\int_0^1 \left[\int_0^\infty (S_k[z]-Q_k)\,\Gamma(z;\hat{x}_k,\{\Delta_{k,j}\}_{j=1}^n,m_k)dz\right] dm_k \cdot \frac{h(\hat{x}'_k)}{h(\hat{x}_k)} = 0.
\end{aligned}$$

Therefore, $\Delta'_{k,j} > \Delta_{k,j}$ for some j. Let $a(k)$ be the first j such that $\Delta'_{k,j} > \Delta_{k,j}$.

First, consider $k=\tau_1$. Note that for $j \notin \mathscr{T}$, since $\hat{x}'_j \leq \hat{x}_j$ and $\hat{x}'_k > \hat{x}_k$, $\Delta'_{k,j} < \Delta_{k,j}$. So, $a(\tau_1) \in \mathscr{T}$ and $a(\tau_1) > \tau_1$. Second, consider $k = a(\tau_1)$. Likewise, $a^{(2)}(\tau_1) = a(a(\tau_1))$ must be in $\mathscr{T}$. By the definition of $a(\tau_1)$, for any $j \in \mathscr{T}$ and $j < a(\tau_1)$, $\Delta'_{\tau_1,j} \leq \Delta_{\tau_1,j}$, and $\Delta'_{\tau_1,a(\tau_1)} > \Delta_{\tau_1,a(\tau_1)}$. So, for these j,

$$\Delta'_{a(\tau_1),j} = \Delta'_{\tau_1,j} - \Delta'_{\tau_1,a(\tau_1)} < \Delta_{\tau_1,j} - \Delta_{\tau_1,a(\tau_1)} = \Delta_{a(\tau_1),j},$$

which implies $a(a(\tau_1)) > a(\tau_1)$. Iterating the procedure, we end up with an infinite sequence $\{a^{(m)}(\tau_1)\}_{m=1}^{+\infty}$ in $\mathscr{T}$. This is impossible because $\mathscr{T}$ is a finite set.

Therefore, the types with $\hat{x}'_k > \hat{x}_k$ do not exist; nor do the types with $\hat{x}'_k < \hat{x}_k$. The solution is unique. Note that the solution is the limits $\{\hat{x}^0_k\}_{k=1}^n$ in Part I.

Part III: The equation system is the necessary and sufficient condition for $\{\hat{x}_k\}_{k=1}^n$ to be the limits of the cutoffs as $\sigma \to 0$.

Suppose as $\sigma \to 0$, $\{\hat{x}^\sigma_k\}_{k=1}^n$ do not converge to $\{\hat{x}^0_k\}_{k=1}^n$. That means, there exists ε and an infinite sequence $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\max_k \left|\hat{x}^{\sigma_m}_k - \hat{x}^0_k\right| > \varepsilon$. According to Part I and Part II, there exists an infinite subsequence $\{\sigma_m\}_{m=1}^{+\infty}$ of $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\{\hat{x}^{\sigma_m}_k\}_{k=1}^n$ converges to $\{\hat{x}^0_k\}_{k=1}^n$, which is impossible. Therefore, as $\sigma \to 0$, $\{\hat{x}^\sigma_k\}_{k=1}^n$ converge to $\{\hat{x}^0_k\}_{k=1}^n$.

Conversely, if $\{\hat{x}_k\}_{k=1}^n$ satisfy the equation system, when the security bundle $\{(S_k,Q_k)\}_{k=1}^n$ is issued, as $\sigma \to 0$, $\{\hat{x}^\sigma_k\}_{k=1}^n$ must converge to the solution of the equation system, which is uniquely $\{\hat{x}_k\}_{k=1}^n$.

Proof of Proposition 3

Property 1

Suppose for $M \in \left(0, \sum_{j=1}^n Q_j\right)$, $\gamma_k(M)$ solves

$$M = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,j}\right).$$

Note that $\gamma_k(0) = 0$.

$$\begin{aligned}
\int_0^M f_k(y)\,dy = & \int_{m_k=\gamma_k(0)}^{\gamma_k(M)} \frac{d(1-m_k)/d\Phi^{-1}(1-m_k)}{-d\left[\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)\right]/d\Phi^{-1}(1-m_k)} \\
& \times d\left[\sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)\right] \\
= & \int_{m_k=\gamma_k(M)}^{\gamma_k(0)} d(1-m_k) \\
= & \gamma_k(M).
\end{aligned}$$

So, Property 1 holds.

Property 2

Since $\gamma_k(M)$ is unique for any k and $M \in \left(0, \sum_{j=1}^n Q_j\right)$, then

$$\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,j} = \Phi^{-1}(1-\gamma_j(M)).$$

Therefore,

$$\begin{aligned}
M = & \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-\gamma_j(M))\right) \\
= & \sum_{j=1}^n Q_j \gamma_j(M) = \int_0^M \sum_{j=1}^n Q_j f_j(y)\,dy,
\end{aligned}$$

Taking derivative with respect to M, we obtain $\sum_{j=1}^n Q_j f_j(M) = 1$.

Property 3

Note that for $j \leq L(k)$, $\Delta_{k,j} = -\infty$. According to Property 1, if $\int_0^M f_k(y)\,dy > 0$,

$$\Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right) = \Phi(-\infty) = 0$$

for $j \leq L(k)$, and

$$\Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,k}\right) = 1 - \int_0^M f_k(y)\,dy < 1,$$

so $M > \sum_{j=1}^{L(k)} Q_j$. If $\int_0^M f_k(y)\,dy = 0$, for $j > L(k)$,

$$\Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right) = \Phi(+\infty) = 1,$$

so $M \leq \sum_{j=1}^{L(k)} Q_j$. Combing the two arguments, we obtain

$$\int_0^M f_k(y)\,dy > 0 \Leftrightarrow M > \sum_{j=1}^{L(k)} Q_j.$$

Likewise, we obtain

$$\int_0^M f_k(y)\,dy < 1 \Leftrightarrow M < \sum_{j=1}^{U(k)} Q_j.$$

For $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, $\int_0^M f_k(y)\,dy \in (0,1)$, Property 1 implies

$$\sum_{j=1}^{U(k)} Q_j - \sum_{j=L(k)+1}^{U(k)} Q_j \Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right) = M.$$

Taking derivative with respect to M,

$$f_k(M) = \frac{\phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right)\right)}{\sum_{j=L(k)+1}^{U(k)} Q_j \phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right)},$$

which is positive because $\Delta_{k,j}$ here are all finite.

Property 4

For $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$,

$$\sum_{j=1}^{U(k)} Q_j - \sum_{j=L(k)+1}^{U(k)} Q_j \Phi\left(\Phi^{-1}\left(1 - \int_0^M f_k(y)\,dy\right) + \Delta_{k,j}\right) = M.$$

It is straightforward to see the left-hand side is strictly increasing in $\int_0^M f_k(y)\,dy$ and strictly decreasing in $\Delta_{k,i}$. So, $\int_0^M f_k(y)\,dy$ is strictly increasing in $\Delta_{k,i}$.

Property 5

Consider $L(k) < i \le U(k)$. Then $L(i) = L(k)$ and $U(i) = U(k)$. Therefore, $f_k(M)$ and $f_i(M)$ are both positive for $M \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$, so their ratio is well defined in the region.

By $\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,i} = \Phi^{-1}(1-\gamma_i(M))$,

$$d\gamma_i(M) = \frac{\phi\left(\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,i}\right)}{\phi\left(\Phi^{-1}(1-\gamma_k(M))\right)} d\gamma_k(M),$$

so

$$\frac{f_k(M)}{f_i(M)} = \frac{d\gamma_k(M)/dM}{d\gamma_i(M)/dM} = \frac{\phi\left(\Phi^{-1}(1-\gamma_k(M))\right)}{\phi\left(\Phi^{-1}(1-\gamma_k(M)) + \Delta_{k,i}\right)}.$$

By SMLRP, we obtain Property 5.

Proof of Proposition 4

In the state θ, mass $Q_k m_k^\sigma(\theta)$ of Type-k agents accept their offers, so the principal's expected payoff is

$$E[\pi^P] = \int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta,$$

where

$$\begin{aligned}\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) &\equiv \int_0^{+\infty} \left(z - \sum_{k=1}^{n} m_k^\sigma(\theta) Q_k s_k[z]\right) g\left(z; \theta, \sum_{k=1}^{n} Q_k m_k^\sigma(\theta\right) dz \\ &= E\left[z|\theta, \sum_{k=1}^{n} Q_k m_k^\sigma(\theta)\right] - \sum_{k=1}^{n} m_k^\sigma(\theta) E\left[S_k[z]|\theta, \sum_{k=1}^{n} Q_k m_k^\sigma(\theta)\right]\end{aligned}$$

For any σ and $\varepsilon > 0$, there exists $t_1 > 0$ such that for any $\theta < \hat{x}_k^\sigma - t_1\sigma$,

$$m_k^\sigma(\theta) = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right) < 1 - \Phi(t_1) < \varepsilon,$$

and for any $\theta > \hat{x}_k^\sigma + t_1\sigma$,

$$m_k^\sigma(\theta) = 1 - \Phi\left(\frac{\hat{x}_k^\sigma - \theta}{\sigma}\right) > 1 - \Phi(-t_1) > 1 - \varepsilon.$$

Consider σ that is sufficiently small such that $\hat{x}_k^\sigma - t_1\sigma > \hat{x}_k - 1$ and $\hat{x}_k^\sigma + t_1\sigma < \hat{x}_k + 1$.

Consider $m_1^\sigma(\theta)$ and $1\{\theta > \hat{x}_1\}$. For $\theta < \underline{\hat{x}_1^\sigma} \equiv \min\{\hat{x}_1^\sigma - t_1\sigma, \hat{x}_1\}$,

$$\begin{aligned}
&\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \\
<&E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] dz \\
\leq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) + E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right],
\end{aligned}$$

so

$$\begin{aligned}
&\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta \\
<&\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \left\{ E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \right\} h(\theta) d\theta.
\end{aligned}$$

For θ satisfying $\sum_{k=2}^n Q_k m_k^\sigma(\theta) - \lambda(\theta) > 0$,

$$\begin{aligned}
E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] &= E_1\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E_1\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \\
&\leq Q_1\varepsilon.
\end{aligned}$$

Similarly, for θ satisfying $\sum_{k=2}^n Q_k m_k^\sigma(\theta) - \lambda(\theta) \leq -Q_1\varepsilon$,

$$E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \leq Q_1\varepsilon.$$

For θ satisfying $-Q_1\varepsilon < \sum_{k=2}^n Q_k m_k^\sigma(\theta) - \lambda(\theta) \leq 0$,

$$E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right]$$

is bounded. Therefore, as $\varepsilon \to 0$,

$$\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \left\{ E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \right\} h(\theta) d\theta$$

can be arbitrarily small.

On the other hand, for $\theta < \underline{\hat{x}_1^\sigma}$,

$$\begin{aligned}
&\Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \\
>&E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \varepsilon E\left[S_1[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] \\
\geq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) - \varepsilon E[z|\theta, 1] - \left\{\sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - \sum_{k=2}^n m_k^\sigma(\theta) E\left[S_k[z]\,|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta\right.\right. \\
\geq&\Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) - \varepsilon E[z|\theta, 1] - \sum_{k=2}^n R_k \left\{E\left[z|\theta, Q_1\varepsilon + \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right] - E\left[z|\theta, \sum_{k=2}^n Q_k m_k^\sigma(\theta)\right]\right\}.
\end{aligned}$$

Following the same argument, we obtain that as $\varepsilon \to 0$,

$$\left|\int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\underline{\hat{x}_1^\sigma}} \Pi(0, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right|$$

can be arbitrarily small.

Likewise, let $\overline{\hat{x}_1^\sigma} \equiv \max\{\hat{x}_1^\sigma + t_1\sigma, \hat{x}_1\}$. As $\varepsilon \to 0$,

$$\left|\int_{\overline{\hat{x}_1^\sigma}}^{+\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{\overline{\hat{x}_1^\sigma}}^{+\infty} \Pi(1\{\theta > \hat{x}_1\}, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right|$$

can be arbitrarily small.

For $\theta \in [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}] \subseteq [\hat{x}_1 - 1, \hat{x}_1 + 1]$,

$$0 \leq \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) \leq E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right],$$

so

$$\begin{aligned}
&\left|\int_{\underline{\hat{x}_1^\sigma}}^{\overline{\hat{x}_1^\sigma}} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{\underline{\hat{x}_1^\sigma}}^{\overline{\hat{x}_1^\sigma}} \Pi(1\{\theta > \hat{x}_1\}, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right| \\
\leq&2E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right] \sup\{h(\cdot)\} \cdot \left(\overline{\hat{x}_1^\sigma} - \underline{\hat{x}_1^\sigma}\right) \\
\leq&2E\left[z|\hat{x}_1 + 1, \sum_{k=1}^n Q_k\right] \sup\{h(\cdot)\} \cdot \left(|\hat{x}_1 - \hat{x}_1^\sigma| + 2t_1\sigma\right),
\end{aligned}$$

which converges to 0 as $\sigma \to 0$.

To sum up, for any $\delta > 0$, there exists $\overline{\sigma}_1$ such that for any $\sigma < \overline{\sigma}_1$,

$$\left|\int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta - \int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta)) h(\theta) d\theta\right| < \delta.$$

Repeating the analysis on all $k \in \{1,2,\ldots,n\}$, we obtain as $\sigma \to 0$,

$$\int_{-\infty}^{\infty} \Pi(m_1^\sigma(\theta), m_2^\sigma(\theta), \ldots, m_n^\sigma(\theta))h(\theta)d\theta \to \int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, 1\{\theta > \hat{x}_2\}, \ldots, 1\{\theta > \hat{x}_n\})h(\theta)d\theta.$$

It is easy to see

$$\begin{aligned}
&\int_{-\infty}^{\infty} \Pi(1\{\theta > \hat{x}_1\}, 1\{\theta > \hat{x}_2\}, \ldots, 1\{\theta > \hat{x}_n\})\lambda(\theta)h(\theta)d\theta \\
&= \sum_{k=0}^{n} \int_{\hat{x}_k}^{\hat{x}_{k+1}} \left[\int_0^{+\infty} \left(z - \sum_{j=1}^{k} Q_j s_j[z] \right) g\left(z; \theta, \sum_{j=1}^{k} Q_j \right) dz \right] h(\theta)d\theta,
\end{aligned}$$

where $\hat{x}_0 = -\infty$ and $\hat{x}_{n+1} = +\infty$.

Next, we show that $\sum_{k=1}^{n} Q_k m_k^\sigma(\theta)$ converges to $\sum_{k=1}^{n} Q_k \cdot 1\{\theta \geq \hat{x}_k\}$ in probability. Consider $m_1^\sigma(\theta)$ and $1\{\theta \geq \hat{x}_1\}$. According to the above construction, for $\theta \notin [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}]$, $\left| m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\} \right| < \varepsilon$, so

$$Pr\left[|m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\}| > \varepsilon \right] \leq Pr\left[\theta \in [\underline{\hat{x}_1^\sigma}, \overline{\hat{x}_1^\sigma}] \right] \leq \sup\{h(\cdot)\} \cdot \left(|\hat{x}_1 - \hat{x}_1^\sigma| + 2t_1\sigma \right).$$

That means, $Pr\left[\left| m_1^\sigma(\theta) - 1\{\theta \geq \hat{x}_1\} \right| > \varepsilon \right]$ can be arbitrarily small when σ is sufficiently small. Similarly, this argument applies to all other k. Therefore,

$$\lim_{\sigma \to 0} Pr\left[\left| \sum_{k=1}^{n} Q_k m_k^\sigma(\theta) - \sum_{k=1}^{n} Q_k \cdot 1\{\theta \geq \hat{x}_k\} \right| > \varepsilon \right] = 0.$$

Proof of Proposition 5

Consider a n-type security bundle that induces multiple cutoffs in the limit case. Suppose that regime change occurs at $\hat{\theta}$, i is the last type whose cutoff is smaller than $\hat{\theta}$, and m is the first type whose cutoff is larger than $\hat{\theta}$. The principal's expected payoff equals

$$\begin{aligned}
&\sum_{k=0}^{i} \int_{\hat{x}_k}^{\hat{x}_{k+1}} \left(E_0\left[z \mid \theta, \sum_{j=1}^{k} Q_j \right] - E_0\left[\sum_{j=1}^{k} S_j[z] \mid \theta, \sum_{j=1}^{k} Q_j \right] \right) h(\theta)d\theta \\
&+ \int_{\hat{\theta}}^{\hat{x}_m} \left(E_1\left[z \mid \theta, \sum_{j=1}^{m-1} Q_j \right] - E_1\left[\sum_{j=1}^{k} S_j[z] \mid \theta, \sum_{j=1}^{m-1} Q_j \right] \right) h(\theta)d\theta \\
&+ \sum_{k=m}^{n} \int_{\hat{x}_k}^{\hat{x}_{k+1}} \left(E_1\left[z \mid \theta, \sum_{j=1}^{k} Q_j \right] - E_1\left[\sum_{j=1}^{k} S_j[z] \mid \theta, \sum_{j=1}^{k} Q_j \right] \right) h(\theta)d\theta.
\end{aligned}$$

Consider that the principal does not offer securities to the agents with cutoffs greater than $\hat{\theta}$.

All other agents still play the same strategy, and the principal's expected payoff equals

$$\sum_{k=0}^{i}\int_{\hat{x}_k}^{\hat{x}_{k+1}}\left(E_0\left[z\mid\theta,\sum_{j=1}^{k}Q_j\right]-E_0\left[\sum_{j=1}^{k}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]\right)h(\theta)d\theta$$
$$+\int_{\hat{\theta}}^{\hat{x}_m}\left(E_1\left[z\mid\theta,\sum_{j=1}^{m-1}Q_j\right]-E_1\left[\sum_{j=1}^{k}S_j[z]\mid\theta,\sum_{j=1}^{m-1}Q_j\right]\right)h(\theta)d\theta$$
$$+\sum_{k=m}^{n}\int_{\hat{x}_k}^{\hat{x}_{k+1}}\left(E_1\left[z\mid\theta,\sum_{j=1}^{m-1}Q_j\right]-E_1\left[\sum_{j=1}^{m-1}S_j[z]\mid\theta,\sum_{j=1}^{m-1}Q_j\right]\right)h(\theta)d\theta.$$

Notice that for $k\geq m$ and $\theta>\hat{x}_k$,

$$E_1\left[z\mid\theta,\sum_{j=1}^{k}Q_j\right]-E_1\left[z\mid\theta,\sum_{j=1}^{m-1}Q_j\right]\leq\sum_{j=m}^{k}Q_j$$

and

$$\begin{aligned}&E_1\left[\sum_{j=1}^{k}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]-E_1\left[\sum_{j=1}^{m-1}S_j[z]\mid\theta,\sum_{j=1}^{m-1}Q_j\right]\\>&E_1\left[\sum_{j=1}^{k}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]-E_1\left[\sum_{j=1}^{m-1}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]\\=&E_1\left[\sum_{j=m}^{k}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]>\sum_{j=m}^{k}Q_j.\end{aligned}$$

The principal becomes strictly better off.

Further, fixing $\{Q_j,\Delta_{k,j}\}_{j=1}^{n}$, the principal offers the first i types $\delta_k\cdot s_k[\cdot]$ such that

$$\int_0^{\infty}\delta_k\cdot s_k[z]\left[\int_0^{\infty}g\left(z;\hat{\theta},M\right)f_k(M)\,dM\right]dz=1.$$

All agents have the same cutoff $\hat{\theta}$, and he principal's expected payoff equals

$$\int_{\hat{\theta}}^{\infty}\left(E_1\left[z\mid\theta,\sum_{j=1}^{m-1}Q_j\right]-E_1\left[\sum_{j=1}^{i}\delta_jS_j[z]+\sum_{j=i+1}^{m-1}S_j[z]\mid\theta,\sum_{j=1}^{m-1}Q_j\right]\right)h(\theta)d\theta.$$

Notice that $\delta_k<1$ and for $k\leq i$ and $\theta>\hat{x}_k$,

$$E_0\left[z\mid\theta,\sum_{j=1}^{k}Q_j\right]-E_0\left[\sum_{j=1}^{k}S_j[z]\mid\theta,\sum_{j=1}^{k}Q_j\right]<0.$$

The principal becomes strictly better off.

Proof of Lemma 3

Denote the alternative security bundle as $\{(S'_k, Q'_k)\}_{k=1}^n$ where

$$S'_k = \begin{cases} S_i + S_{i+1} & \text{if } k = i \\ 0 & \text{if } k = i+1 \\ S_k, & \text{otherwise} \end{cases}$$

and

$$Q'_k = \begin{cases} Q_i + Q_{i+1} & \text{if } k = i \\ 0 & \text{if } k = i+1 \\ Q_k, & \text{otherwise} \end{cases} .$$

$(S'_{i+1}, Q'_{i+1}) = (0,0)$ is used only for notational convenience and can be ignored. We only need to check that the condition in Proposition 2 holds for the alternative security bundle with $\{\hat{x}_k, \Delta_{j,k}\}_{j,k \in \{1,2,\ldots,n\}}$. Since $\Delta_{i,i+1} = 0$, $\Delta_{i,k} = \Delta_{i+1,k}$ and $\hat{x}_i = \hat{x}_{i+1}$.

$$\begin{aligned} f\left(M; \{Q'_j, \Delta_{k,j}\}_{j=1}^n\right) &= \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q'_j \phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)} \\ &= \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j \neq i, j \neq i+1} Q_j \phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right) + (Q_i + Q_{i+1}) \phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,i}\right)} \\ &= \frac{\phi\left(\Phi^{-1}(1-m_k)\right)}{\sum_{j=1}^n Q_j \phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right)}, \end{aligned}$$

where

$$M = \sum_{j=1}^n Q'_j - \sum_{j=1}^n Q'_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right) = \sum_{j=1}^n Q_j - \sum_{j=1}^n Q_j \Phi\left(\Phi^{-1}(1-m_k) + \Delta_{k,j}\right).$$

So,

$$f\left(M; \{Q'_j, \Delta_{k,j}\}_{j=1}^n\right) = f\left(M; \{Q_j, \Delta_{k,j}\}_{j=1}^n\right).$$

That means, for k other than i or $i+1$,

$$\begin{aligned}
&\int_0^\infty S'_k[z]\left[\int_0^\infty g(z;\hat{x}_k,M)\,f\left(M;\{Q'_j,\Delta_{k,j}\}_{j=1}^n\right)dM\right]dz\\
=&\int_0^\infty S_k[z]\left[\int_0^\infty g(z;\hat{x}_k,M)\,f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^n\right)dM\right]dz\\
=&Q_k
\end{aligned}$$

For $k=i$,

$$\begin{aligned}
&\int_0^\infty S'_i[z]\left[\int_0^\infty g(z;\hat{x}_i,M)\,f\left(M;\{Q'_j,\Delta_{i,j}\}_{j=1}^n\right)dM\right]dz\\
=&\int_0^\infty \left(S_i[z]+S_{i+1}[z]\right)\left[\int_0^\infty g(z;\hat{x}_i,M)\,f\left(M;\{Q_j,\Delta_{i,j}\}_{j=1}^n\right)dM\right]dz\\
=&Q_i+Q_{i+1}
\end{aligned}$$

Therefore, for all types, the condition in Proposition 2 holds.

The proof of the converse is similar.

Proof of Lemma 4

Consider $W^P(z)/W_k^A(z)$.

$$\begin{aligned}
\frac{W^P(z)}{W_k^A(z)} &= \int_{\hat{\theta}}^\infty \frac{g(z;\theta,K)}{\int_0^\infty g\left(z;\hat{\theta},M\right)f_k(M)\,dM}h(\theta)d\theta\\
&= \int_{\hat{\theta}}^\infty \frac{1}{\int_{\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j}\frac{g\left(z;\hat{\theta},M\right)}{g(z;\theta,K)}f_k(M)\,dM}h(\theta)d\theta.
\end{aligned}$$

Since $\hat{\theta}<\theta$ and $M<K$, $g\left(z;\hat{\theta},M\right)/g(z;\theta,K)$ is strictly decreasing in z. Therefore $W^P(z)/W_k^A(z)$ is strictly increasing in z.

Consider $W_k^A(z)/W_{k-1}^A(z)$. If $\Delta_{k-1,k}=+\infty$, $U(k-1)=L(k)$.

$$\frac{W_k^A(z)}{W_{k-1}^A(z)} = \frac{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},M\right) f_k\left(M\right) dM}{\int_{\sum_{j=1}^{L(k-1)} Q_j}^{\sum_{j=1}^{U(k-1)} Q_j} g\left(z;\hat{\theta},y\right) f_{k-1}\left(y\right) dy}$$

$$= \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(M\right)}{\int_{\sum_{j=1}^{L(k-1)} Q_j}^{\sum_{j=1}^{U(k-1)} Q_j} \frac{g\left(z;\hat{\theta},y\right)}{g\left(z;\hat{\theta},M\right)} f_{k-1}\left(y\right) dy} dM.$$

Since $M > y$, $g\left(z;\hat{\theta},y\right)/g\left(z;\hat{\theta},M\right)$ is strictly decreasing in z. Therefore, $W_k^A(z)/W_{k-1}^A(z)$ is strictly increasing in z.

If $\Delta_{k-1,k} < +\infty$, let

$$\Omega(z,y) \equiv \int_{\sum_{j=1}^{L(k)} Q_j}^{y} \frac{g\left(z;\hat{\theta},M\right) \cdot f_{k-1}\left(M\right)}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},M\right) \cdot f_{k-1}\left(M\right) dM} dM.$$

$\Omega(z,y)$ is strictly decreasing in z for any $y \in \left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$ because

$$\Omega(z,y) = \frac{1}{1 + \frac{\int_{y}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},M\right) \cdot f_{k-1}(M) dM}{\int_{\sum_{j=1}^{L(k)} Q_j}^{y} g\left(z;\hat{\theta},M\right) \cdot f_{k-1}(M) dM}}.$$

$$\frac{W_k^A(z)}{W_{k-1}^A(z)} = \frac{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},y\right) \cdot f_k\left(y\right) dy}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},M\right) \cdot f_{k-1}\left(M\right) dM}$$

$$= \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(y\right)}{f_{k-1}\left(y\right)} \frac{g\left(z;\hat{\theta},y\right) f_{k-1}\left(y\right)}{\int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} g\left(z;\hat{\theta},M\right) \cdot f_{k-1}\left(M\right) dM} dy$$

$$= \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \frac{f_k\left(y\right)}{f_{k-1}\left(y\right)} d\Omega(z,y).$$

By $\Omega\left(z, \sum_{j=1}^{L(k)} Q_j\right) = 0$ and $\Omega\left(z, \sum_{j=1}^{U(k)} Q_j\right) = 1$, using integration by parts, we obtain

$$\frac{W_k^A(z)}{W_{k-1}^A(z)} = \frac{f_k\left(\sum_{j=1}^{U(k)} Q_j\right)}{f_{k-1}\left(\sum_{j=1}^{U(k)} Q_j\right)} - \int_{\sum_{j=1}^{L(k)} Q_j}^{\sum_{j=1}^{U(k)} Q_j} \Omega(z,y) d\left[\frac{f_k(y)}{f_{k-1}(y)}\right].$$

By the fifth property in Proposition 3, $f_k(y)/f_{k-1}(y)$ is strictly increasing in y and $\Omega(z,y)$ is strictly decreasing in z over $\left(\sum_{j=1}^{L(k)} Q_j, \sum_{j=1}^{U(k)} Q_j\right)$. So, $W_k^A(z)/W_{k-1}^A(z)$ is strictly increasing in z.

Proof of Proposition 6

In this proof, we always take $\{(\hat{x}_k, Q_k, \Delta_{k-1,k})\}_{k=1}^n$ as given and only alter the securities $\{S_k\}_{k=1}^n$. Hence, all agents' perception of participation stays unchanged. Without loss of generality

First, consider the first two types. The expected payment to Type-1 and Type-2 agents is

$$\int_{\hat{\theta}}^{+\infty} E\left[\sum_{k=1}^{2} S_k[z] \mid \theta, K\right] h(\theta) d\theta$$

Let $S_1'[z] \equiv \min\{S_1[z] + S_2[z], F'\}$ and $S_2'[z] \equiv S_1[z] + S_2[z] - S_1'[z]$, where F' is the minimum value such that

$$\int_0^\infty S_1'[z] W_1^A(z) dz = Q_1. \tag{21}$$

I show

$$\int_0^\infty S_2'[z] W_2^A(z) dz > \int_0^\infty S_2[z] W_2^A(z) dz.$$

Since

$$\int_0^\infty S_1[z] W_1^A(z) dz = Q_1,$$

there must exists a minimum $\tilde{z} \geq 0$ such that $S_1[z] - S_1'[z]$ is nonpositive for $z \leq \tilde{z}$ and nonnegative for $z > \tilde{z}$. So,

$$\begin{aligned} &\int_0^{\tilde{z}} \left(S_1'[z] - S_1[z]\right) W_1^A(z) dz \\ =&\int_{\tilde{z}}^{+\infty} \left(S_1[z] - S_1'[z]\right) W_1^A(z) dz. \end{aligned}$$

Then

$$\begin{aligned}
&\int_0^{\infty} S_2'[z]W_2^A(z)dz - \int_0^{\infty} S_2[z]W_2^A(z)dz \\
=&\int_0^{\infty} \left(S_1[z]-S_1'[z]\right) W_2^A(z)dz \\
=&\int_0^{\tilde{z}} \left(S_1[z]-S_1'[z]\right) \frac{W_2^A(z)}{W_1^A(z)} W_1^A(z)dz + \int_{\tilde{z}}^{+\infty} \left(S_1[z]-S_1'[z]\right) \frac{W_2^A(z)}{W_1^A(z)} W_1^A(z)dz.
\end{aligned}$$

Since $W_2^A(z)/W_1^A(z)$ is strictly increasing,

$$\begin{aligned}
&\int_0^{\infty} S_2'[z]W_2^A(z)dz - \int_0^{\infty} S_2[z]W_2^A(z)dz \\
\geq &\int_0^{\tilde{z}} \left(S_1[z]-S_1'[z]\right) \frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})} W_1^A(z)dz + \int_{\tilde{z}}^{+\infty} \left(S_1[z]-S_1'[z]\right) \frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})} W_1^A(z)dz \\
=&\frac{W_2^A(\tilde{z})}{W_1^A(\tilde{z})} \int_0^{+\infty} \left(S_1[z]-S_1'[z]\right) W_1^A(z)dz = 0
\end{aligned}$$

Next, I show

$$\int_{\hat{x}_1}^{\hat{x}_2} \left[\int_0^{+\infty} S_1'[z]g\left(z;\theta,K_1\right)dz\right] h(\theta)d\theta \leq \int_{\hat{x}_1}^{\hat{x}_2} \left[\int_0^{+\infty} S_1[z]g\left(z;\theta,K_1\right)dz\right] h(\theta)d\theta.$$

If $\hat{x}_1 = \hat{x}_2$, it is obvious. If $\hat{x}_1 < \hat{x}_2$, then $\hat{x}_1 = \theta_1$ and $\hat{x}_2 = \theta_2$. The inequality is equivalent to

$$\int_0^{+\infty} S_1'[z]W_1^P(z)dz \leq \int_0^{+\infty} S_1[z]W_1^P(z)dz.$$

Since $W_1^P(z)/W_1^A(z)$ is strictly increasing, we prove it following the same logic as above.

Let

$$\rho_2 = \frac{\int_0^{\infty} S_2'[z]W_2^A(z)dz}{\int_0^{\infty} S_2[z]W_2^A(z)dz}$$

such that

$$\int_0^{\infty} \rho_2 S_2'[z]W_2^A(z)dz = Q_2.$$

Then $\rho_2 \leq 1$. So, the principal can replace S_1 and S_2 with S_1' and $\rho_2 S_2'$ to implement the participation scheme. Offering them, the expected payment to Type-1 and Type-2 agents is lower, i.e.,

$$\begin{aligned}
&\int_{\hat{x}_1}^{\hat{x}_2}\left[\int_0^{+\infty} S_1'[z]g\left(z;\theta,K_1\right)dz\right]h(\theta)d\theta+\int_{\hat{x}_2}^{+\infty}\left[\int_0^{+\infty}\left(S_1'+\rho_2 S_2'\right)[z]g\left(z;\theta,K\left(\theta\right)\right)dz\right]h(\theta)d\theta \\
\leq &\int_{\hat{x}_1}^{\hat{x}_2}\left[\int_0^{+\infty} S_1[z]g\left(z;\theta,K_1\right)dz\right]h(\theta)d\theta+\int_{\hat{x}_2}^{+\infty}\left[\int_0^{+\infty}\left(S_1+S_2\right)[z]g\left(z;\theta,K\left(\theta\right)\right)dz\right]h(\theta)d\theta.
\end{aligned}$$

Since the expected payment to other types of agents does not change, the total expected payment is lower.

Second, consider S_1' and S_3. Likewise, we construct $S_1''[z]\equiv\min\{S_1'[z]+S_3[z],F''\}$, $S_3'[z]\equiv S_1'[z]+S_3[z]-S_1''[z]$, and ρ_3. Following the above analysis, we can show that the total expected payment is strictly lower if the principal offers S_1'' and $\rho_3 S_3$ instead of S_1' and S_3. If $F''>F'$, then $S_1''[z]\geq S_1'[z]$ for a positive measure of z, so

$$\int_0^{\infty} S_1''[z]W_1^A(z)dz>\int_0^{\infty} S_1'[z]W_1^A(z)dz=Q_1.$$

Contradiction! So, $F''\leq F'$ and $S_1''[z]=\min\{\sum_{k=1}^3 S_k[z],F''\}$. Iterating this procedure with all remaining contracts, we end up with $S_1'''[z]\equiv\min\{\sum_{k=1}^n S_k[z],F'''\}$ and a strictly lower total expected payment.

Third, consider T_{F_1}, where F_1 is the minimum value such that

$$\int_0^{\infty} T_{F_1}[z]W_1^A(z)dz=Q_1.$$

Following the above analysis, we know $F_1\leq F'''$, so it is feasible to the offer T_{F_1} to Type-1 agents without changing the contracts to other types. The total expected payment is strictly lower if the principal offers T_{F_1} to Type-1 agents instead of S_1'''.

Fourth, given that the first k contracts take the form $T_{F_k}-T_{F_{k-1}}$ where $0=F_0<F_1<\ldots<F_k$, iterating the above three steps on the $(k+1)$-th contract, we can show that the total expected payment is lower if it takes the form $T_{F_{k+1}}-T_{F_k}$ where $F_{k+1}>F_k$. Finally, we end up with a tranching structure. Moreover, from the above proof, we can see that if such a tranching structure has a strictly lower total expected payment than any other security bundle does.

Proof of Proposition 7

Part I: constructing an alternative security bundle

Suppose that Proposition 7 does not hold. That means, there exists an optimal security bundle with n types $(n<N)$. We pick the **first** one of them and split it into two types in the way described in

Lemma 3 such that all agents use the same strategy as before. Note that this $(n+1)$-type security bundle should also be the optimal. Denote it as $\{(S_j,Q_j)\}_{j=1}^{n+1}$ and its resultant equilibrium as $\{(\hat{x}_j,\Delta_{j-1,j})\}_{j=1}^{n+1}$. In the rest of the proof, $\{(S_j,Q_j)\}_{j=1}^{n+1}$ is referred to as the original security bundle. We intend to show that the original security bundle cannot be the optimal. To this end, it is without loss of generality to assume that this security bundle contains a tranching structure as in Proposition 6. So, $S_j = T_{F_j} - T_{F_{j-1}}$ where $0 = F_0 < F_1 < \ldots < F_{n+1}$.

Since the first type and the second type are the two created by the split, $\hat{x}_1 = \hat{x}_2$, $\Delta_{1,2} = 0$, $L(1) = L(2) = 0$, and $U(1) = U(2)$. Keeping all $\hat{x}_j$ unchanged, consider an alternative equilibrium $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$ where $\eta \in [0,+\infty)$ and

$$\Delta'_{j-1,j} = \begin{cases} +\infty, & \text{if } j=1 \\ \Delta_{1,2}+\eta, & \text{if } j=2 \\ \Delta_{j-1,j}, & \text{if } j\geq 3 \end{cases}.$$

It is easy to see that this equilibrium satisfies Equation (15). Then $\forall k \neq 1$,

$$\Delta'_{k,j} = \begin{cases} \Delta_{k,j}, & \text{if } j\neq 1 \\ \Delta_{k,1}-\eta, & \text{if } j=1 \end{cases},$$

and

$$\Delta'_{1,j} = \begin{cases} \Delta_{1,j}+\eta, & \text{if } j\neq 1 \\ 0, & \text{if } j=1 \end{cases}.$$

Since η is finite, $L(j)$ and $U(j)$ remain the same for all j in the alternative equilibrium. Keeping all Q_j unchanged, consider an alternative security bundle $\{(S'_j,Q_j)\}_{j=1}^{n+1}$.

$$S'_j = \begin{cases} T_{F'_1}, & \text{if } j=1 \\ T_{F'_2}-T_{F'_1}, & \text{if } j=2 \\ \rho_j S_j, & \text{if } j\geq 3 \end{cases}.$$

We claim that if $\{(S'_j,Q_j)\}_{j=1}^{n+1}$ results in the equilibrium $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$, all ρ_j must be weakly smaller than 1. Since $\{(S_j,Q_j)\}_{j=1}^{n+1}$ results in $\{(\hat{x}_j,\Delta_{j-1,j})\}_{j=1}^{n+1}$, by the breakeven condition of the marginal Type-k agents Equation (16), we obtain

$$\int_0^\infty S_k[z]\left[\int_0^\infty g(z;\hat{x}_k,M)\,f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)dM\right]dz = Q_k$$

$$\Leftrightarrow \int_0^\infty S_k[z]\left\{\int_{M=\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j} g(z;\hat{x}_k,M)\,d\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)dy\right]\right\}dz = Q_k$$

$$\Leftrightarrow \int_0^\infty S_k[z]\left\{g\left(z;\hat{x}_k,\sum_{j=1}^{U(k)}Q_j\right)-\int_{M=\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)dy\right]dg(z;\hat{x}_k,M)\right\}dz = Q_k$$

Consider $\{(S'_j,Q_j)\}_{j=1}^{n+1}$ that results in $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$. For $k\geq 3$, the breakeven condition of the marginal Type-k agents implies

$$\rho_k\int_0^\infty S_k[z]\left\{g\left(z;\hat{x}_k,\sum_{j=1}^{U(k)}Q_j\right)-\int_{M=\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta'_{k,j}\}_{j=1}^{n+1}\right)dy\right]dg(z;\hat{x}_k,M)\right\}dz = Q_k.$$

Since $\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)dy$ is weakly increasing in $\Delta_{k,j}$ and

$$\Delta'_{k,j}=\begin{cases}\Delta_{k,j}, & \text{if } j\neq 1\\ \Delta_{k,1}-\eta, & \text{if } j=1\end{cases},$$

$$\int_{M=\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta'_{k,j}\}_{j=1}^{n+1}\right)dy\right]dg(z;\hat{x}_k,M)$$
$$\leq\int_{M=\sum_{j=1}^{L(k)}Q_j}^{\sum_{j=1}^{U(k)}Q_j}\left[\int_0^M f\left(y;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)dy\right]dg(z;\hat{x}_k,M),$$

So $\rho_k\leq 1$.

If there exists a bundle that can result in the equilibrium $\{(\hat{x}_j,\Delta'_{j-1,j})\}_{j=1}^{n+1}$ with F'_2 strictly smaller than F_2, then this bundle can implement the participation scheme at a strictly lower cost than the original optimal one. In the second part of the proof, we focus on F'_1 and F'_2.

Part II: there exists positive Q_1 and η such that $F'_2 < F_2$.

Note that the original bundle is created by splitting the first type of the optimal contract into two. This implies that Q_1+Q_2 and all Q_k for $k\geq 3$ are fixed. So is $\sum_{j=1}^{U(1)}Q_j$. Hence, we have two choice variables, Q_1 and η, for constructing the alternative bundle. We denote the PPs of the first types as $f_1(M;Q_1,\eta)$ and $f_2(M;Q_1,\eta)$ respectively. Particularly, at $\eta=0$,

$$f_1(M;Q_1,0)=f_2(M;Q_1,0)=f\left(M;\{Q_j,\Delta_{k,j}\}_{j=1}^{n+1}\right)$$

does not vary with Q_1, which follows Lemma 3.

According to the breakeven condition of the marginal Type-1 agents Equation (16), we have

$$\begin{aligned} Q_1 = &\int_0^{F_1'} (z-F_0)\left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1,M)\, f_1(M;Q_1,\eta)\, dM\right]\cdot dz \\ &+\int_{F_1'}^{+\infty} \left(F_1'-F_0\right)\left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1,M)\, f_1(M;Q_1,\eta)\, dM\right]\cdot dz. \end{aligned} \tag{22}$$

Note that at $\eta=0$, $F_1'=F_1$. Holding Q_1 fixed and taking derivative of the breakeven condition at $\eta=0$, we obtain

$$\begin{aligned} &\frac{dF_1'}{d\eta}\left[1-\int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\, f_1(M;Q_1,0)\, dM\right] \\ =&-\int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_1(y;Q_1,0)\, dy}{d\eta}\cdot d\left[\int_0^{F_1} G(z;\hat{x}_1,M)\, dz\right]. \end{aligned}$$

Likewise, for the marginal Type-2 agents,

$$\begin{aligned} Q_2 = &\int_{F_1'}^{F_2'} \left(z-F_1'\right)\left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1,M)\, f_2(M;Q_1,\eta)\, dM\right]\cdot dz \\ &+\int_{F_2'}^{+\infty} \left(F_2'-F_1'\right)\left[\int_0^{\Sigma_{j=1}^{U(1)} Q_j} g(z;\hat{x}_1,M)\, f_2(M;Q_1,\eta)\, dM\right]\cdot dz. \end{aligned}$$

Note that at $\eta=0$, $F_2'=F_2$, and $f_1(M;Q_1,0)=f_2(M;Q_1,0)$. Holding Q_2 fixed and taking derivative of the breakeven condition at $\eta=0$, we obtain

$$\begin{aligned} &\frac{dF_2'}{d\eta}\left[1-\int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_2;\hat{x}_1,M)\, f_1(M;Q_1,0)\, dM\right] \\ =&\frac{dF_1'}{d\eta}\left[1-\int_0^{\Sigma_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\, f_1(M;Q_1,0)\, dM\right]-\int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_2(y;Q_1,0)\, dy}{d\eta}\cdot d\left[\int_{F_1}^{F_2} G(z;\hat{x}_1,M)\, dz\right] \\ =&-\int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_1(y;Q_1,0)\, dy}{d\eta}\cdot d\left[\int_0^{F_1} G(z;\hat{x}_1,M)\, dz\right]+\int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d\int_0^M f_2(y;Q_1,0)\, dy}{d\eta}\cdot d\left[\int_{F_1}^{F_2} G(z;\hat{x}_1,M)\, dz\right]. \end{aligned}$$

To quantify $d\int_0^M f_1(y;Q_1,0)\,dy/d\eta$, we resort to the first property in Proposition 3 and obtain

$$\sum_{j=1}^{n+1} Q_j - \sum_{j=2}^{n+1} Q_j \Phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}+\eta\right) - Q_1\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right) = M$$

and

$$\begin{aligned}&\sum_{j=1}^{n+1} Q_j - \sum_{j=2}^{n+1} Q_j \Phi\left(\Phi^{-1}\left(1-\int_0^M f_2(y;Q_1,0)\,dy\right)+\Delta_{2,j}\right)\\ &-Q_1\Phi\left(\Phi^{-1}\left(1-\int_0^M f_2(y;Q_1,0)\,dy\right)+\Delta_{2,1}-\eta\right) = M.\end{aligned}$$

Taking derivative with respect to η at $\eta = 0$, we have

$$\begin{aligned}&\left[\sum_{j=3}^{n+1} Q_j \frac{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right)} + Q_1 + Q_2\right]\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\\ =&\sum_{j=2}^{n+1} Q_j \phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right) > 0\end{aligned}$$

and

$$\begin{aligned}&\left[\sum_{j=3}^{n+1} Q_j \frac{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right)} + Q_1 + Q_2\right]\frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\frac{1}{Q_1}\\ =&-\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)\right) < 0.\end{aligned}$$

Here we use $\Delta_{1,2} = 0$ and thus $\int_0^M f_1(y;Q_1,0)\,dy = \int_0^M f_2(y;Q_1,0)\,dy$ at $\eta = 0$.

We claim that there exists Q_1 such that $dF_2'/d\eta < 0$. According to 22, it is easy to see that at $\eta = 0$, F_1 goes to 0 as Q_1 goes to 0, and

$$dQ_1 = dF_1\left[1-\int_0^{\sum_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\,f_1(M;Q_1,0)\,dM\right].$$

By L'Hospital's rule,

$$\lim_{Q_1\to 0}\frac{F_1}{Q_1} = \lim_{Q_1\to 0}\frac{dF_1}{dQ_1} = \lim_{F_1\to 0}\frac{1}{1-\int_0^{\sum_{j=1}^{U(1)} Q_j} G(F_1;\hat{x}_1,M)\,f_1(M;Q_1,0)\,dM} = 1.$$

Since $\int_0^M f_1(y;Q_1,0)\,dy$ does not vary with Q_1, so is $\frac{d\int_0^M f_2(y;Q_1,0)dy}{d\eta}\frac{1}{Q_1}$. Since $G(z;\hat{x}_1,M)$ is decreasing in M for any z,

$$\begin{aligned}&\lim_{Q_1\to 0}\frac{1}{Q_1}\int_{M=0}^{\Sigma_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_{F_1}^{F_2}G(z;\hat{x}_1,M)\,dz\right]\\ =&\int_{M=0}^{\Sigma_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_2(y;Q_1,0)\,dy}{d\eta}\frac{1}{Q_1}\cdot d\left[\int_0^{F_2}G(z;\hat{x}_1,M)\,dz\right]\\ >&0.\end{aligned}$$

We only need to show that

$$\lim_{Q_1\to 0}\frac{1}{Q_1}\int_{M=0}^{\Sigma_{j=1}^{U(1)}Q_j}\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\cdot d\left[\int_0^{F_1}G(z;\hat{x}_1,M)\,dz\right]=0.$$

Consider $M(F)$ such that

$$G(F;\hat{x}_1,M(F))=M(F).$$

It is straightforward to see that $M(F)$ is strictly increasing in F. Since $G(0;\hat{x}_1,M)=0$ for any M, $\lim_{F\to 0}M(F)=0$. Note that

$$\frac{d\int_0^M f_1(y;Q_1,0)\,dy}{d\eta}\leq\frac{\sum_{j=2}^{n+1}Q_j\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)}{Q_1+Q_2}.$$

According to Lemma 6, $\phi(\cdot)$ is bounded, and

$$\lim_{M\to 0}\phi\left(\Phi^{-1}\left(1-\int_0^M f_1(y;Q_1,0)\,dy\right)+\Delta_{1,j}\right)=\lim_{\varepsilon\to+\infty}\phi(\varepsilon)=0.$$

Therefore, for any $\overline{M}$, there exists finite $\chi(\overline{M})$ such that $d\int_0^M f_1(y;Q_1,0)\,dy/d\eta<\chi(\overline{M})$ for any

$M \in [0, \overline{M}]$ and $\chi(\overline{M})$ converges 0 as $\overline{M}$ goes to 0. Then

$$\begin{aligned}
&\left| \frac{1}{Q_1} \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_1(y; Q_1, 0)\, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \right| \\
\leq& \left| \frac{1}{Q_1} \int_{M=0}^{M(F_1)} \frac{d \int_0^M f_1(y; Q_1, 0)\, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \right| + \left| \frac{1}{Q_1} \int_{M=M(F_1)}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_1(y; Q_1, 0)\, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \right| \\
\leq& \left| \frac{1}{Q_1} \int_{M=0}^{M(F_1)} \chi(M(F_1)) \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \right| + \left| \frac{1}{Q_1} \int_{M=M(F_1)}^{\Sigma_{j=1}^{U(1)} Q_j} \chi\left(\sum_{j=1}^{U(1)} Q_j \right) \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \right| \\
\leq& \frac{\chi(M(F_1))}{Q_1} \left| \int_0^{F_1} G(z; \hat{x}_1, 0)\, dz - \int_0^{F_1} G(z; \hat{x}_1, M(F_1))\, dz \right| \\
&+ \frac{\chi\left(\Sigma_{j=1}^{U(1)} Q_j\right)}{Q_1} \left| \int_0^{F_1} G(z; \hat{x}_1, M(F_1))\, dz - \int_0^{F_1} G\left(z; \hat{x}_1, \sum_{j=1}^{U(1)} Q_j\right) dz \right| \\
\leq& \frac{\chi(M(F_1))}{Q_1} \left| \int_0^{F_1} G(z; \hat{x}_1, 0)\, dz \right| + \frac{\chi\left(\Sigma_{j=1}^{U(1)} Q_j\right)}{Q_1} \left| \int_0^{F_1} G(z; \hat{x}_1, M(F_1))\, dz \right| \\
\leq& \frac{F_1}{Q_1} \chi(M(F_1)) + \frac{F_1}{Q_1} \chi\left(\sum_{j=1}^{U(1)} Q_j \right) G(F_1; \hat{x}_1, M(F_1)) \\
=& \frac{F_1}{Q_1} \chi(M(F_1)) + \frac{F_1}{Q_1} \chi\left(\sum_{j=1}^{U(1)} Q_j \right) M(F_1).
\end{aligned}$$

As $Q_1 \to 0$, $F_1/Q_1 \to 1$ and $F_1 \to 0$, so $M(F_1) \to 0$. We obtain

$$\frac{1}{Q_1} \int_{M=0}^{\Sigma_{j=1}^{U(1)} Q_j} \frac{d \int_0^M f_1(y; Q_1, 0)\, dy}{d\eta} \cdot d\left[\int_0^{F_1} G(z; \hat{x}_1, M)\, dz \right] \to 0.$$

We confirm that there exists Q_1 such that $dF_2'/d\eta < 0$.

To sum up, we prove that there exists positive Q_1 and η such that $F_2' < F_2$. With such Q_1 and η, $\{(S_j', Q_j)\}_{j=1}^{n+1}$ implements the same participation scheme as $\{(S_j, Q_j)\}_{j=1}^{n+1}$ but its total expected payment to the agents is strictly lower. Hence, $\{(S_j, Q_j)\}_{j=1}^{n+1}$ cannot be optimal, and the optimal security bundle must have N types.

Proof of Proposition 8

It is not hard to see that Proposition 5 holds even if the principal can only offer collinear securities. Without loss of generality, suppose the firm offers $\{(Q_k s_k, Q_k)\}_{k=1}^{n}$ and all types of agents have the same cutoff $\hat{\theta}$. Then according to Proposition 2,

$$\int_0^\infty s[z] \left[\int_0^\infty g\left(z; \hat{\theta}, M\right) f_k(M)\, dM \right] dz = p_k.$$

Multiplying by Q_k and summing over k from 1 to n,

$$\int_0^\infty s[z]\left[\int_0^\infty g\left(z;\hat{\theta},M\right)\sum_{k=1}^n Q_k f_k(M)\,dM\right]dz=\sum_{k=1}^n Q_k p_k.$$

According to the second property and the third property in Proposition 3, $\sum_{k=1}^n Q_k f_k(M)=1$ for $M\in(0,\sum_{k=1}^n Q_k)$, so

$$\int_0^\infty s[z]\left[\int_0^{\sum_{k=1}^n Q_k} g\left(z;\hat{\theta},M\right)dM\right]dz=\sum_{k=1}^n Q_k p_k.$$

Note that the aggregate security is

$$S[\cdot]=\sum_{k=1}^n \frac{Q_k}{p_k}s[\cdot].$$

Then

$$\int_0^\infty S[z]\left[\int_0^{\sum_{k=1}^n Q_k} g\left(z;\hat{\theta},M\right)dM\right]dz=\sum_{k=1}^n \frac{Q_k}{p_k}\cdot\sum_{k=1}^n Q_k p_k.$$

Let

$$\delta=\frac{\left(\sum_{k=1}^n Q_k\right)^2}{\sum_{k=1}^n \frac{Q_k}{p_k}\cdot\sum_{k=1}^n Q_k p_k}.$$

When p_k are not all equal, by Cauchy-Schwarz Inequality,

$$\sum_{k=1}^n \frac{Q_k}{p_k}\cdot\sum_{k=1}^n Q_k p_k>\left(\sum_{k=1}^n Q_k\right)^2,$$

so $\delta<1$. Consider that the principal offers $\delta/\sum_{k=1}^n Q_k\cdot S[\cdot]$ to each agent instead. Then

$$\int_0^\infty \delta/\sum_{k=1}^n Q_k\cdot S[z]\left[\int_0^{\sum_{k=1}^n Q_k} g\left(z;\hat{\theta},M\right)\frac{1}{\sum_{k=1}^n Q_k}dM\right]dz=1.$$

In this case, all these agents have the same perception of participation, which is a uniform distribution over $(0,\sum_{k=1}^n Q_k)$. Hence, their breakeven conditions are satisfied at $\hat{\theta}$. That means, this alternative security bundle can induce the same mass of agents to have the same common cutoff but at a strictly lower expected cost. Proposition 8 is proved.

Proof of Proposition 9

Part I: $\tilde{V}_t(x) > x$ is a necessary condition for zero premium.

Suppose $K\left(\theta;\{(K_t,\theta_t)\}_{t=1}^T\right)$ can be implemented by a finite-type security bundle with zero premium and there exist $\hat{t} \in \{1,2,\dots,T\}$ and $\hat{x} \in [0, K_{\hat{t}} - K_{\hat{t}-1})$ such that $\tilde{V}_{\hat{t}}(\hat{x}) \leq \hat{x}$, i.e.,

$$V\left(\theta_{\hat{t}}, \hat{x} + K_{\hat{t}-1}\right) \leq \hat{x} + K_{\hat{t}-1}$$

Suppose the security bundle is $\{(S_k, Q_k)\}_{k=1}^n$. Consider the k-th type with the cutoff $\theta_{\hat{t}}$ such that $\sum_{j=1}^{k-1} Q_j \leq \hat{x} + K_{\hat{t}-1} < \sum_{j=1}^{k} Q_j$. On the one hand, because of zero premium,

$$V\left(\theta_{\hat{t}}, \sum_{j=1}^{L(k)} Q_j\right) \geq \sum_{j=1}^{k} Q_j.$$

.On the other hand, since $L(k) \leq k-1$ and $\tilde{V}_t(x)$ is increasing in x,

$$V\left(\theta_{\hat{t}}, \sum_{j=1}^{L(k)} Q_j\right) \leq V\left(\theta_{\hat{t}}, \sum_{j=1}^{k-1} Q_j\right) \leq V\left(\theta_{\hat{t}}, \hat{x} + K_{t-1}\right) \leq \hat{x} + K_{\hat{t}-1}.$$

So,

$$\sum_{j=1}^{k} Q_j \leq = \hat{x} + K_{\hat{t}-1}.$$

Contradiction! Therefore, such $\hat{t}$ and $\hat{x}$ cannot exist.

Part II: $\tilde{V}_t(x) > x$ is a sufficient condition for the participation scheme to have an n^*-type security bundle achieving zero premium.

Suppose that $\tilde{V}_t(x) > x$ for any $t \in \{1,2,\dots,T\}$ and $x \in (0, K_t - K_{t-1})$.

First, we prove the existence of n_t^*. Suppose not. Then $\tilde{V}_t^{(n)}(0) < K_t - K_{t-1}$ for any n. Since $\tilde{V}_t(x)/x$ is continuous over $[\tilde{V}_t(0), K_t - K_{t-1}]$, there exists $\underline{x} \in [\tilde{V}_t(0), K_t - K_{t-1}]$ such that $\tilde{V}_t(x)/x \geq \tilde{V}_t(\underline{x})/\underline{x}$. Notice

$$\frac{\tilde{V}_t^{(n)}(0)}{K_t - K_{t-1}} = \frac{\tilde{V}_t(0)}{K_t - K_{t-1}} \prod_{k=2}^{n} \frac{\tilde{V}_t\left(\tilde{V}_t^{(k-1)}(0)\right)}{\tilde{V}_t^{(k-1)}(0)}$$

and $\tilde{V}_t^{(k-1)}(0) \in [\tilde{V}_t(0), K_t - K_{t-1}]$. So,

$$\frac{\tilde{V}_t^{(n)}(0)}{K_t - K_{t-1}} \geq \frac{\tilde{V}_t(0)}{K_t - K_{t-1}} \left[\frac{\tilde{V}_t(\underline{x})}{\underline{x}}\right]^{n-1}.$$

Since $\tilde{V}_t(\underline{x})/\underline{x} > 1$, when n is sufficiently large, $\tilde{V}_t^{(n)}(0) > K_t - K_{t-1}$. Contradiction!

Second, we construct an n^*-type security bundle achieving zero premium. For the cutoff θ_t, the n_t^* contracts are represented by

$$\left\{\left(S_{k+\sum_{i=1}^{t-1} n_i^*}, Q_{k+\sum_{i=1}^{t-1} n_i^*}\right)\right\}_{k=1}^{n_t^*}$$

where for $k < n_i^*$

$$\begin{aligned} S_{k+\sum_{i=1}^{t-1} n_i^*} &= T_{\tilde{V}_t^{(k)}(0)+K_{t-1}} - T_{\tilde{V}_t^{(k-1)}(0)+K_{t-1}} \\ Q_{k+\sum_{i=1}^{t-1} n_i^*} &= \tilde{V}_t^{(k)}(0) - \tilde{V}_t^{(k-1)}(0), \end{aligned}$$

and

$$\begin{aligned} S_{\sum_{i=1}^{t} n_i^*} &= T_{K_t} - T_{\tilde{V}_t^{(n_t^*-1)}(0)+K_{t-1}} \\ Q_{\sum_{i=1}^{t} n_i^*} &= K_t - K_{t-1} - \tilde{V}_t^{(n_t^*-1)}(0). \end{aligned}$$

It is straightforward to see this n^*-type security bundle can implement the participation scheme with zero premium and all $\Delta_{k-1,k}$ being $+\infty$.

Part III: any security bundle with fewer than n^* types cannot achieve zero premium.

Suppose $\{(S_k, Q_k)\}_{k=1}^{n}$ with $\Delta_{k-1,k} > 0$ can achieve zero premium and there are l types with the cutoff θ_l: $\tau+1, \tau+2, \ldots, \tau+l$. According to Proposition 6, it is without loss of generality to assume $S_k \equiv T_{F_k} - T_{F_{k-1}}$. Because of zero premium,

$$\begin{aligned} \sum_{j=\tau+1}^{\tau+k} Q_j &\leq V\left(\theta_t, \sum_{j=\tau+1}^{\tau+k-1} Q_j + K_{t-1}\right) - K_{t-1} \\ &= \tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+k-1} Q_j\right). \end{aligned}$$

Since $\tilde{V}_t(x)$ is increasing in x,

$$K_t - K_{t-1} = \sum_{j=\tau+1}^{\tau+l} Q_j \leq \tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+l-1} Q_j\right) \leq \tilde{V}_t\left(\tilde{V}_t\left(\sum_{j=\tau+1}^{\tau+l-2} Q_j\right)\right) = \tilde{V}_i^{(l)}(0).$$

By the definition of n_i^*, $l \geq n_i^*$. Therefore, to achieve zero premium, the whole bundle must have at least n^* types.